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Multi-attribute decision-making using Archimedean aggregation operator in T-spherical fuzzy environment

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ABSTRACT

As an extension of several fuzzy structures such as fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, q-rung ortho-pair fuzzy sets, and T-spherical fuzzy (TSF) sets (TSFSs), are an effective tool for controlling the vagueness of data. Archimedean t-conorm (ATCN) and t-norm (ATN), which consists of the t-conorm (TCN) and t-norm (TN) families, is a crucial approach for fuzzy sets to produce extensive operational rules. In this manuscript, for TSF numbers some core operational laws are initiated based on ATCN and ATN, also some basic characteristics of these operational laws are investigated. Secondly, based on these operational laws TSF Archimedean weighted averaging (TSPFAWA) and TSF Archimedean weighted geometric (TSPFAWG) operators are initiated. Thirdly, we investigated special cases of these aggregation operators and some basic properties. On the behalf of the TSPFAWA and TSPFAWG operators, a novel method for solving multiple attribute decision-making (MADM) problems using TSF information is also devised. Lastly, a numerical example is provided to demonstrate the applicability of the suggested technique, and a comparison analysis is done to demonstrate its advantages.

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1. Introduction

Uncertainty always exists in almost all types of information that are based on human options. To reduce this uncertainty (Zadeh 1965), announced the notion of the fuzzy set (FS) with membership degree (MD). In addition, Atanassov (1989) offered the thought of the intuitionistic fuzzy (IF) set (IFS) with MD and nonmember ship degree (NMD) always lies between the range of $[0,1]$. IFS provides greater freedom to study imprecise and ambiguous information. Furthermore, Yager (2013), provided the awareness of the Pythagorean fuzzy set (PyFS) their sum of the square of its NMD and MD lies between $[0,1]$. The generalization of the PyFS set is provided by Yager (2016), in the form of taking q^{th} power of MD and NMD is called a q-rung ortho pair fuzzy set (q-ROFS).

Human opinion has also a certain degree of refusal and abstinence. This indicates that the previous structures of the generalized form of FS are unable to solve these types of confusing problems. To cover these types of problems, a Picture fuzzy set (PFS) with the involvement of abstinence degree (AD) along with MD and NMD firstly defined by Cuong (2015). The range of PFS also lies between the interval $[0,1]$. Later on, Mahmood et al. (2019), presented that the generalized form of PFS is said to be spherical FS (SFS), and after some modification, by taking the q^{th} power of SFS he introduced a new idea called the T-spherical FS (TSFS).

In FS theory MADM approach is a trending technique nowadays for the aggregation of information. We use the MADM approach for the selection of the most suitable option from the given list of multiple options by under consideration of the multiple standards at the same time. In FS theory number of AOs based on TN and TCNs was defined by several mathematicians. For example, Munir et al. (2020), examined the Einstein interactive AOs of TSFS. Liu and Wang (2018) proposed the concepts of geometric and averaging AOs for orthopair FS and also examined their application to MADM problems. Wei (2010), presented the IF trapezoidal fuzzy operators, and Jana et al. (2019) proposed the concept of Dombi AOs in IFS theory as well as in PyFS theory. The q-ROFS Aczel-Alsina AOs defined by Khan et al. (2022). Ullah, Mahmood, and Garg (2020) provided the thought of Hamacher AOs in the TSF environment while Hamacher AOs in the IFSs system based on entropy measurement developed by Garg (2019). In FS theory IF hybrid geometric and arithmetic, AOs were proposed by Ye (2017). The idea of Maclaurin symmetric AOs proposed by Ullah (2021). The IF cubic AOs developed and their applicability to the MADM problem were discussed by Kaur and Garg (2018), and Liu et al. (2019) discussed power Muirhead Maen AOs for TSFS and their application to the MADM problems. In addition to that, some applications of TSFS theory by utilizing several types of operators and measures are summaries by Ullah (2021).

In the theory of probabilistic fuzzy metric space Menger (1942), firstly proposed the concept of the triangular norm. Over time, many TN and their corresponding TCNs and their operations are discussed in the FS theory. For example, Strict ATCN and ATN were discussed by Nguyen, Kreinovich, and Wojciechowski (1998). IF integral-based ATCN and ATN were presented by Lei et al. (2016). Continuous integral-based ATN and ATCNs were introduced by Ai et al. (2020). The idea of maximal discrete ATN and ATCNs was presented by (Bejines and Navara 2022). The interactive ATN and ATCN were presented by Wang and Garg (2021).

Almost all the AOs are developed on the bases of operational laws of algebra like the product, sum, scalar multiplication, and power operation. However, ATN and ATCN is the generalized form of many kinds of FS theoretic operations. ATN and ATCN provide better understanding and preciseness in results than other present TN and TCN in the FS system. Because in ATN and ATCN have obtained all AOs by changing the value of the function and it covers a wide class of TN and TCNs like algebraic, Frank, Hamacher, Einstein, and many other classes of TN and TCNs. So, AOs based on ATN and ATCN are more significant than other existing AOs.

For medical science problems, the MADM approach is a useful technique, such as the smart medical devices selection for diagnosing the problem in the human body based on IF Choquet integral by Büyüközkan and Göçer (2019). The selection of LASER surgical instruments for surgery by using the MADM technique based on neutrosophic FS and fuzzy TOPSIS method by Farooq and Saqlain (2021). Pre-operative surgical tool ordering by utilizing the MADM approach by Miller et al. (2008), and risk evaluation in the selection of the prioritization of the medical devices by Jamshidi et al. (2015). Pamucar et al. (2022) delivered the concept of supplier selection of healthcare instruments during the covid-19 pandemic situation. An efficient surgery instruments supply selection methodology to hospital pharmacy by using fuzzy the MADM approach by Manivel and Ranganathan (2019). By using the MADM algorithm to measure the effect of air quality on the surgical instruments in operation theater by Colella et al. (2022). Rahman and Lee (2013), by using fuzzy logic studied the assessment of the disturbance of surgical instruments during surgery. Tian and Juan proposed the method of selection of the best surgical instruments through manipulation and perception. By using interval-valued IF model evaluation of surgical instruments risk during organ transplant by Salimian et al. (2022).

A surgical instrument is a tool or apparatus used during an operation or surgery to carry out particular tasks or achieve desired results such as modification of the biological tissues. Many surgical instruments have been invented by several manufacturing companies. It is very difficult to decide for the hospital management departments for giving the tender to the surgical company which offers the best quality instruments at reasonable price rates. In this critical situation, we have proposed an ATSFWA and ATSFWG AOs model for dealing with this kind of problem. In this article, we have considered the alternatives such as nature of the material, purity of the material, accuracy in the functionality, designed by computer numerical control (CNC) machines or by hand for the selection, and using the proposed ATSFWA and ATSFWG AOs algorithm for the aggregating the fuzzy information.

The article offers the following information: Section 2, discussed some fundamental definitions for a better understanding of the article. In section 3. we introduce operational laws for the aggregation of TSF information. Section 4, proposed new AOs like ATSFWA, Archimedean TSF ordered weighted averaging (ATSFOWA), and Archimedean TSF hybrid weighted averaging (ATSFOWA). ATSFWG, Archimedean TSF ordered weighted geometric (ATSFOWG), Archimedean TSF hybrid weighted geometric (ATSFOWG). Section 5 also discussed some desirable axioms. In section 6, develop the algorithm for explaining the MADM problematic issue. On the behalf of our proposed AOs, solve the MADM problem in section 7. Compare our aggregation outcomes with other existing AOs in section 8. Section 9, discussed the advantages of this article. Section 10, provided the comprehensive conclusion and future research interest mentioned.

2. Preliminaries

2.1. T-Spherical Fuzzy Set

In this segment, we discussed a few fundamental concepts related to TSFS, TN, and TCN also their operational laws. These key concepts will make our article easy to understand for the reader.

Definition 1: (Mahmood et al., 2019), Consider the TSFS on X is $T = \{\langle x, \rho_T(x), \varphi_T(x), \tau_T(x)\rangle\}$ and ρ_T, φ_T and τ_T are represents the MD, AD and NMG of $x \in X$ respectively lies between the range $[0,1]$ and $0 \leq Sum(\rho_T^t, \varphi_T^t, \tau_T^t) \leq 1$ for $n \in \mathbb{Z}$. Here $\sqrt[t]{1 - Sum\big(\rho_T(x), \varphi_T(x), \tau_T(x)\big)}$ is the refusal grade. So, this triplet $(\rho_T^t, \varphi_T^t, \tau_T^t)$ is called TSF number (TSFN).

Remark 1 In the light of the above definition 1. We have proved that TSFS is comprehensive form of FS as compare to the other existing fuzzy structures. That is,

i. When we take $t = 2$ in the above structure then TSFS becomes SFS.

ii. When we take $t = 1$ in the above structure then TSFS becomes PFS.

iii. When we take $t = 2$ and $\varphi = 0$ in the above structure then TSFS becomes PyFS.

iv. When we take $t = 1$ and $\varphi = 0$ in the above structure then TSFS becomes IFS.

v. When we take $t = 2$ and $\varphi = \tau = 0$ in the above structure then TSFS becomes FS.

Definition 2: (Mahmood et al., 2019), Consider any TSFN $T = \{\langle x, \rho(x), \varphi(x), \tau(x)\rangle\}$, for TSFS score function (SF) is defined as follows:

$$S(T) = \rho_T^t - \varphi_T^t - \tau_T^t \tag{1}$$

Where $S(T) \in [-1, 1]$.

Consider any TSFN such as $T = \{\langle x, \rho(x), \varphi(x), \tau(x)\rangle\}$, then for TSFS score function (SF) can be defined as follows:

$$A(T) = \rho_T^t + \varphi_T^t + \tau_T^t \tag{2}$$

where $A(T) \in [0, 1]$.

Definition 3: (Mahmood et al., 2019), Consider T_1 and T_2 be two TSFNs, then the ordering of those TSFNs by using the follows the principles:

i. If $S(\mathrm{T}_1) > S(T_2)$, then $\mathrm{T}_1 > \mathrm{T}_2$

ii. If $S(\mathrm{T}_1) > S(\mathrm{T}_2)$, then

iii. If $A(\mathrm{T}_1) > A(\mathrm{T}_2)$, then $\mathrm{T}_1 > \mathrm{T}_2$

iv. If $A(\mathrm{T}_1) = A(\mathrm{T}_2)$, then $\mathrm{T}_1 \approx \mathrm{T}_2$.

Definition 4: (Mahmood et al., 2019), Let three TSFNs such as $R = (\rho, \varphi, \tau), R_1 = (\rho_1, \varphi_1, \tau_1)$ and $R_2 = (\rho_2, \varphi_2, \tau_2)$ and $\lambda > 0$, the following four fundamental algebraic operations can be defined as given below:

i. $R_1 \oplus R_2 = \left(\sqrt[t]{\rho_1^t + \rho_2^t - \rho_1^t\rho_2^t}, \varphi_1\varphi_2, \tau_1\tau_2\right)$

ii. $R_1 \otimes R_2 = \left(\rho_1\rho_2, \varphi_1\varphi_2, \sqrt[t]{\tau_1^t + \tau_2^t - \tau_1^t\tau_2^t}\right)$

iii. $\lambda R = \left(\sqrt[t]{1 - (1 - \rho^t)^\lambda}, \varphi_R^\lambda, \tau_R^\lambda\right); \lambda > 0$

iv. $R^\lambda = \left(\rho_R^\lambda, \varphi_R^\lambda, \sqrt[t]{1 - (1 - \tau^t)^\lambda}\right); \lambda > 0$

2.2 Archimedean TN and Archimedean TCN

ATCN and ATN were introduced by Klir and Yuan (1995).

Definition 5:(Wang and Garg 2021), In FS theory TN is a function $\mathfrak{Q}: [0,1] \times [0,1] \to [0,1]$ that must be satisfies the given following axioms $\forall\ \mathfrak{a}, \mathfrak{b}, \mathfrak{d} \in [0,1]$.

Property $\mathfrak{Q}1$. $\mathfrak{Q}(\mathfrak{a}, 0) = \mathfrak{a}\ \forall \mathfrak{a}$.

Property $\mathfrak{Q}2$. If $\mathfrak{b} \leq \mathfrak{b}^*$ and $\mathfrak{d} \leq \mathfrak{d}^*$ then $\mathfrak{Q}(\mathfrak{b}, \mathfrak{d}) \leq \mathfrak{Q}(\mathfrak{b}^*, \mathfrak{d}^*)$.

Property $\mathfrak{Q}3$. $\mathfrak{Q}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{Q}(\mathfrak{b}, \mathfrak{a})\ \forall \mathfrak{a}\mathfrak{b}$.

Property $\mathfrak{Q}4$. $\mathfrak{Q}\big(\mathfrak{a}, \mathfrak{Q}(\mathfrak{b}, \mathfrak{d})\big) = \mathfrak{Q}(\mathfrak{Q}(\mathfrak{a}, \mathfrak{b}), \mathfrak{d})\ \forall \mathfrak{a}\mathfrak{b}\mathfrak{d}$.

Definition 6: (Wang and Garg 2021), In the FS theory, TCN is such type of function defined as $\mathfrak{R}: [0,1] \times [0,1] \to [0,1]$ that satisfies the following properties $\forall\ \mathfrak{a}, \mathfrak{b}, \mathfrak{d} \in [0,1]$.

Property $\mathfrak{R}1$. $\mathfrak{R}(\mathfrak{a}, 1) = \mathfrak{a}\ \forall \mathfrak{a}$.

Property $\mathfrak{R}2$. If $\mathfrak{b} \leq \mathfrak{b}^*$ and $\mathfrak{d} \leq \mathfrak{d}^*$ then $\mathfrak{R}(\mathfrak{b}, \mathfrak{d}) \leq \mathfrak{R}(\mathfrak{b}^*, \mathfrak{d}^*)$.

Property $\mathfrak{R}3$. $\mathfrak{R}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{R}(\mathfrak{b}, \mathfrak{a})\ \forall \mathfrak{a}\mathfrak{b}$.

Property $\mathfrak{R}4$. $\mathfrak{R}\big(\mathfrak{a}, \mathfrak{Q}(\mathfrak{b}, \mathfrak{d})\big) = \mathfrak{R}(\mathfrak{R}(\mathfrak{a}, \mathfrak{b}), \mathfrak{d})\ \forall \mathfrak{a}\mathfrak{b}\mathfrak{d}$.

Definition 7: (Wang and Garg, 2021), In FS theory TCN is such type of function $\mathfrak{Q}(\mathfrak{a},\mathfrak{b})$ is said to be ATCN if it is continuous and $\mathfrak{Q}(\mathfrak{a},\mathfrak{a}) > \mathfrak{a}\ \forall \mathfrak{a} \in [0,1]$. If each variable for $\mathfrak{ab} \in [0,1]$ is strictly increasing then ATCN is called strict ATCN.

Definition 8: (Wang and Garg, 2021), In FS theory TN is a function $\mathfrak{R}(\mathfrak{a},\mathfrak{b})$ is said to be ATN if it is continuous and $\mathfrak{R}(\mathfrak{a},\mathfrak{a}) < \mathfrak{a}\ \forall \mathfrak{a} \in [0,1]$. If each variable for $\mathfrak{ab} \in [0,1]$ is strictly increasing then ATN is called the strict ATN.

Definition 9: On $\mathbb{R}$ a continues function $\mathcal{H}$ from interval $[0,1]$ is defined as, if $\mathcal{H}$ is a strictly decreasing function and $\mathcal{H}(1) = 0$, then $\mathcal{H}$ is said to be a strictly decreasing generator.

Definition 10: On $\mathbb{R}$ a continues function $\mathcal{K}$ from interval $[0,1]$ is defined as, if $\mathcal{K}$ is always a strictly increasing function and $\mathcal{K}(0) = 0$, then $\mathcal{K}$ is said to be a strictly increasing generator.

Definition 11: (Klement and Mesiar 2005), An increasing generator $\mathscr{y}$ for a strict ATCN is stated as

$$\mathfrak{Q}(\mathfrak{a},\mathfrak{b}) = \mathscr{y}^{-1}\big(\mathscr{y}(\mathfrak{a}) + \mathscr{y}(\mathfrak{b})\big) \text{ with } \mathscr{y}(r) = \mathfrak{z}(1-r)\ \forall\ \mathfrak{a},\mathfrak{b},r \in [0,1] \quad (3)$$

And similarly, a decreasing generator $\mathfrak{z}$ for ATN is expressed as

$$\mathfrak{R}(\mathfrak{a},\mathfrak{b}) = \mathfrak{z}^{-1}\ (\mathfrak{z}(\mathfrak{a}) + \mathfrak{z}(\mathfrak{b}))\ \forall\ \mathfrak{a},\mathfrak{b} \in [0,1] \quad (4)$$

Klement and Mesiar (2005), presented some TN and TCN for such following functions which are given below, such as:

1. Consider $\mathfrak{z}(r) = -logr$, then $\mathscr{y}(r) = \mathfrak{z}(1-r) = -\log(1-r)$, $\mathfrak{z}^{-1}(r) = e^{-r}$, $\mathscr{y}(r) = 1 - e^{-r}$ then TCN can be expressed as $\mathfrak{Q}^{\mathcal{E}}(\mathfrak{a},\mathfrak{b}) = \mathfrak{a} + \mathfrak{b} - \mathfrak{ab}$ and TN $\mathfrak{R}^{\mathcal{E}}(\mathfrak{a},\mathfrak{b}) = \mathfrak{ab}$.
2. Consider $\mathfrak{z}(r) = -\log\left(\frac{(2-r)}{r}\right)$, then $\mathscr{y}(r) = -\log\left(\frac{(2-(1-r))}{1-r}\right)$, $\mathfrak{z}^{-1}(r) = \frac{2}{(e^r+1)}$, $\mathscr{y}^{-1}(r) = 1 - \frac{2}{(e^r+1)}$ then Einstein's TCN can be expressed as $\mathfrak{Q}^{\mathcal{E}}(\mathfrak{a},\mathfrak{b}) = \frac{(\mathfrak{a}+\mathfrak{b})}{(1+\mathfrak{ab})}$ and $\mathfrak{R}^{\mathcal{E}}(\mathfrak{a},\mathfrak{b}) = \frac{\mathfrak{ab}}{(1+(1-\mathfrak{a})(1-\mathfrak{b}))}$.
3. Consider $\mathfrak{z}(r) = \left(\frac{((\theta+(1-\theta))r)}{r}\right)$, $\theta > 0$, then $\mathscr{y}(r) = \left(\frac{(\theta+(1-\theta)(1-r))}{(1-r)}\right)$, $\mathfrak{z}^{-1}(r) = \left(\frac{\theta}{(e^r+\theta-1)}\right)$, $\mathscr{y}^{-1}(r) = \left(\frac{\theta}{e^r+\theta-1}\right)$ then Hamacher TCN can be expressed as $\mathfrak{Q}^{\mathcal{H}}_{\theta}(\mathfrak{a},\mathfrak{b}) = \left(\frac{(\mathfrak{a}+\mathfrak{b}-\mathfrak{ab}-(1-\theta)\mathfrak{ab})}{(1-(1-\theta)\mathfrak{ab})}\right)$ and TN $\mathfrak{R}^{\mathcal{H}}_{\theta}(\mathfrak{a},\mathfrak{b}) = \frac{\mathfrak{ab}}{(\theta+(1-\theta)(\mathfrak{a}+\mathfrak{b}-\mathfrak{ab}))}$, $\theta > 0$. If we take $\theta = 1$, then Hamacher TCN and TN can be transformed into algebraic TCN and TN respectively. If we take $\theta = 2$, then Hamacher TCN and TN are reduced into the Einstein TCN and TN respectively.
4. Consider $\mathfrak{z}(r) = \log\left(\frac{\theta-1}{\theta^r-1}\right)$, $\theta > 0$, then $\mathscr{y}(r) = \left(\frac{\theta-1}{\theta^{1-r}-1}\right)$, $\mathfrak{z}^{-1}(r) = \frac{\log((\theta-1+e^r)/e^r)}{\log\theta}$, $\mathscr{y}^{-1}(r) = 1 - \left(\frac{(\theta-1+e^r)/e^r}{\log\theta}\right)$, then Frank TCN and TN can be expressed as $\mathfrak{Q}^{\mathcal{F}}_{\theta}(\mathfrak{a},\mathfrak{b}) = 1 - \log_{\theta}\left(1 + \frac{(\theta^{1-\mathfrak{a}}-1)(\theta^{1-\mathfrak{b}}-1)}{\theta-1}\right)$, $\mathfrak{R}^{\mathcal{F}}_{\theta} = \log_{\theta}\left(1 + \frac{(\theta^{\mathfrak{a}}-1)(\theta^{\mathfrak{b}}-1)}{\theta-1}\right)$, $\theta > 0$. If $\theta \to 0$ then $\lim_{n\to\infty} \mathfrak{z}(r) = -\log r$.

3. Operational laws

3.1Archimedean TCN and TN on TSFNs

In the FS theory, ATCN and ATN play a vital role in the aggregation of fuzzy data and in solving MADM problems (Garg and Arora, 2021). In this segment, on the behalf of ATCN and ATN, we define some basic operational rules for the TSF environment as follows:

Definition 12: Let three TSFNs such as $R = (\rho,\varphi,\tau)$, $R_1 = (\rho_1,\varphi_1,\tau_1)$ and $R_2 = (\rho_2,\varphi_2,\tau_2)$ and $\lambda > 0$, now on the bases of TSFNs we defined some new operations for ATCN and ATN as follows:

1. $R_1 \oplus R_2 = \left(\sqrt{\mathfrak{Q}((\rho_1^t)^2,(\rho_2^t)^2)}, \sqrt{\mathfrak{R}((\varphi_1^t)^2,(\varphi_2^t)^2)}, \sqrt{\mathfrak{R}((\tau_1^t)^2,(\tau_2^t)^2)}\right)$
$$= \left(\sqrt{\mathscr{y}^{-1}(\mathscr{y}(\rho_1^t)^2,\mathscr{y}(\rho_2^t)^2)}, \sqrt{\mathfrak{z}^{-1}(\mathfrak{z}(\varphi_1^t)^2,\mathfrak{z}(\varphi_2^t)^2)}, \sqrt{\mathfrak{z}^{-1}(\mathfrak{z}(\tau_1^t)^2,\mathfrak{z}(\tau_2^t)^2)}\right)$$
2. $R_1 \otimes R_2 = \left(\left(\sqrt{\mathfrak{R}((\rho_1^t)^2,(\rho_2^t)^2)}, \sqrt{\mathfrak{Q}((\varphi_1^t)^2,(\varphi_2^t)^2)}, \sqrt{\mathfrak{Q}((\tau_1^t)^2,(\tau_2^t)^2)}\right)\right)$
$$= \left(\sqrt{\mathfrak{z}^{-1}(\mathfrak{z}(\rho_1^t)^2,\mathfrak{z}(\rho_2^t)^2)}, \sqrt{\mathscr{y}^{-1}(\mathscr{y}(\varphi_1^t)^2,\mathscr{y}(\varphi_2^t)^2)}, \sqrt{\mathscr{y}^{-1}(\mathscr{y}(\tau_1^t)^2,\mathscr{y}(\tau_2^t)^2)}\right)$$
3. $\lambda R = \left(\sqrt{\mathscr{y}^{-1}(\lambda\mathscr{y}(\rho^t)^2)}, \sqrt{\mathfrak{z}^{-1}(\lambda\mathfrak{z}(\varphi^t)^2)}, \sqrt{\mathfrak{z}^{-1}(\lambda\mathfrak{z}(\tau^t)^2)}\right)$
4. $R^{\lambda} = \left(\sqrt{\mathfrak{z}^{-1}(\lambda\mathfrak{z}(\rho^t)^2)}, \sqrt{\mathscr{y}^{-1}(\lambda\mathscr{y}(\varphi^t)^2)}, \sqrt{\mathscr{y}^{-1}(\lambda\mathscr{y}(\tau^t)^2)}\right)$

- **(Algebraic)** (Nguyen, Walker, and Walker, 2018), When $\mathfrak{z}(r) = -\log r$ then operational laws are defined in the Definition 4. Are found.
- **(Einstein)** When $\mathfrak{z}(r) = \log\left(\frac{(2-r)}{r}\right)$, then the operational laws are defined as follows:

1. $R_1 \oplus R_2 = \left(\sqrt{\frac{\left((\rho_1^t)^2+(\rho_2^t)^2\right)}{1-(\rho_1^t)^2(\rho_2^t)^2}}, \frac{\varphi_1^t\varphi_2^t}{\sqrt{1+\left(1-(\varphi_1^t)^2\right)\left(1-(\varphi_2^t)^2\right)}}, \frac{\varphi_1^t\varphi_2^t}{\sqrt{1+\left(1-(\tau_1^t)^2\right)\left(1-(\tau_2^t)^2\right)}}\right)$
2. $R_1 \otimes R_2 = \left(\frac{\rho_1^t\rho_2^t}{\sqrt{1+\left(1-(\rho_1^t)^2\right)\left(1-(\rho_2^t)^2\right)}}, \sqrt{\frac{\left((\varphi_1^t)^2+(\varphi_2^t)^2\right)}{1-(\varphi_1^t)^2(\varphi_2^t)^2}}, \sqrt{\frac{\left((\tau_1^t)^2+(\tau_2^t)^2\right)}{1-(\tau_1^t)^2(\tau_2^t)^2}}\right)$
3. $\lambda R = \left(\sqrt{\frac{(1+(\rho^t)^2)^\lambda-(1-(\rho^t)^2)^\lambda}{(1+(\rho^t)^2)^\lambda+(1-(\rho^t)^2)^\lambda}}, \frac{\sqrt{2}(\varphi^t)^\lambda}{\sqrt{((2-(\varphi^t)^2)^\lambda+((\varphi^t)^2)^\lambda)}}, \frac{\sqrt{2}(\tau^t)^\lambda}{\sqrt{((2-(\tau^t)^2)^\lambda+((\tau^t)^2)^\lambda)}}\right), \lambda > 0$
4. $R^\lambda = \left(\frac{\sqrt{2}(\rho^t)^\lambda}{\sqrt{((2-(\rho^t)^2)^\lambda+((\rho^t)^2)^\lambda)}}, \sqrt{\frac{(1+(\varphi^t)^2)^\lambda-(1-(\varphi^t)^2)^\lambda}{(1+(\varphi^t)^2)^\lambda+(1-(\varphi^t)^2)^\lambda}}, \sqrt{\frac{(1+(\tau^t)^2)^\lambda-(1-(\tau^t)^2)^\lambda}{(1+(\tau^t)^2)^\lambda+(1-(\tau^t)^2)^\lambda}}\right)$

These are Einstein's operational laws on TSFs.

- **(Hamacher)** When $\mathfrak{z}(r) = \log\left(\frac{(\theta+(1-\theta)r)}{r}\right), \theta > 0, \lambda > 0$, the following operational laws are defined as

1. $R_1 \oplus R_2 =$
$$\left(\sqrt{\frac{(\rho_1^t)^2+(\rho_2^t)^2-(\rho_1^t)^2(\rho_2^t)^2-(1-\theta)(\rho_1^t)^2(\rho_2^t)^2}{1-(1-\theta)(\rho_1^t)^2(\rho_2^t)^2}}, \frac{\varphi_1^t\varphi_2^t}{\sqrt{\theta+(1-\theta)\left((\varphi_1^t)^2+(\varphi_2^t)^2-(\varphi_1^t)^2(\varphi_2^t)^2\right)}}, \frac{\tau_1^t\tau_2^t}{\sqrt{\theta+(1-\theta)\left((\tau_1^t)^2+(\tau_2^t)^2-(\tau_1^t)^2(\tau_2^t)^2\right)}}\right)$$
2. $R_1 \otimes R_2 =$
$$\left(\frac{\rho_1^t\rho_2^t}{\sqrt{\theta+(1-\theta)\left((\rho_1^t)^2+(\rho_2^t)^2-(\rho_1^t)^2(\rho_2^t)^2\right)}}, \sqrt{\frac{(\varphi_1^t)^2+(\varphi_2^t)^2-(\varphi_1^t)^2(\varphi_2^t)^2-(1-\theta)(\varphi_1^t)^2(\varphi_2^t)^2}{1-(1-\theta)(\varphi_1^t)^2(\varphi_2^t)^2}}, \sqrt{\frac{(\tau_1^t)^2+(\tau_2^t)^2-(\tau_1^t)^2(\tau_2^t)^2-(1-\theta)(\tau_1^t)^2(\tau_2^t)^2}{1-(1-\theta)(\tau_1^t)^2(\tau_2^t)^2}}\right)$$
3. $\lambda R = \left(\sqrt{\frac{(1+(\theta-1)(\rho^t)^2)^\lambda-(1-(\rho^t)^2)^\lambda}{(1+(\theta-1)(\rho^t)^2)^\lambda+(1-\theta)(1-(\rho^t)^2)^\lambda}}, \frac{\sqrt{\theta}(\varphi^t)^\lambda}{\sqrt{(1+(\theta-1)(1-(\varphi^t)^2)^\lambda+(\theta-1)(\varphi^t)^{2\lambda})}}, \frac{\sqrt{\theta}(\tau^t)^\lambda}{\sqrt{(1+(\theta-1)(1-(\tau^t)^2)^\lambda(\theta-1)(\tau^t)^{2\lambda})}}\right)$
4. $R^\lambda = \left(\frac{\sqrt{\theta}(\rho^t)^\lambda}{\sqrt{(1+(\theta-1)(1-(\rho^t)^2)^\lambda+(\theta-1)(\rho^t)^{2\lambda})}}, \sqrt{\frac{(1+(\theta-1)(\varphi^t)^2)^\lambda-(1-(\varphi^t)^2)^\lambda}{(1+(\theta-1)(\varphi^t)^2)^\lambda+(1-\theta)(1-(\varphi^t)^2)^\lambda}}, \sqrt{\frac{(1+(\theta-1)(\tau^t)^2)^\lambda-(1-(\tau^t)^2)^\lambda}{(1+(\theta-1)(\tau^t)^2)^\lambda+(1-\theta)(1-(\tau^t)^2)^\lambda}}\right)$

These are the Hamacher operational laws on TSFs.

- **(Frank)** When $\mathfrak{z}(r) = \left(\frac{\theta-1}{\theta^r-1}\right), \theta > 0$, and $\lambda > 0$, then the following operational laws are defined as follows:

1. $R_1 \oplus R_2 =$
$$\left(\sqrt{1-\log_\theta\left(1+\frac{\left(\theta^{1-(\rho_1^t)^2}-1\right)\left(\theta^{1-(\rho_2^t)^2}-1\right)}{\theta-1}\right)}, \sqrt{\log_\theta\left(1+\frac{\left(\theta^{(\varphi_1^t)^2}-1\right)\left(\theta^{(\varphi_2^t)^2}-1\right)}{\theta-1}\right)}, \sqrt{\log_\theta\left(1+\frac{\left(\theta^{(\tau_1^t)^2}-1\right)\left(\theta^{(\tau_2^t)^2}-1\right)}{\theta-1}\right)}\right)$$
2. $R_1 \otimes R_2 = \left(\begin{array}{c}\sqrt{\log_\theta\left(1+\frac{\left(\theta^{(\rho_1^t)^2}-1\right)\left(\theta^{(\rho_2^t)^2}-1\right)}{\theta-1}\right)}, \sqrt{1-\log_\theta\left(1+\frac{\left(\theta^{1-(\varphi_1^t)^2}-1\right)\left(\theta^{1-(\varphi_2^t)^2}-1\right)}{\theta-1}\right)}, \\ \sqrt{1-\log_\theta\left(1+\frac{\left(\theta^{1-(\tau_1^t)^2}-1\right)\left(\theta^{1-(\tau_2^t)^2}-1\right)}{\theta-1}\right)}\end{array}\right)$

3. $$\lambda R=\left(\sqrt{1-\log_{\theta}\left(1+\frac{\left(\theta^{1-(\rho^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)},\sqrt{\log_{\theta}\left(1+\frac{\left(\theta^{(\varphi^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)},\sqrt{\log_{\theta}\left(1+\frac{\left(\theta^{(\tau^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)}\right)$$

4. $$R^{\lambda}=\left(\sqrt{\log_{\theta}\left(1+\frac{\left(\theta^{(\rho^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)},\sqrt{1-\log_{\theta}\left(1+\frac{\left(\theta^{1-(\varphi^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)},\sqrt{1-\log_{\theta}\left(1+\frac{\left(\theta^{1-(\tau^{t})^{2}}-1\right)^{\lambda}}{(\theta-1)^{\lambda-1}}\right)}\right)$$

These are the Frank operational laws in TSFs

4. T-Spherical Fuzzy Archimedean Weighted Averaging AOs

This part, of the article, develops the ATSFWA, ATSFOWA, and ATSFHWA operators and discussed their fundamental characteristics in detail.

Definition 13: Let $R_{\mathrm{j}}\ (\mathrm{j}=1,2,\dots,n)\in R$ be the set of TSFNs, and $\mathfrak{h}'=(\mathfrak{h}'_1,\mathfrak{h}'_2,\dots,\mathfrak{h}'_n)$ be considered as weight vectors (WVs) with conditions $\mathfrak{h}'_{\mathrm{j}}\in[0,1]$ and $\sum_{\mathrm{j}=1}^{n}\mathfrak{h}'_{\mathrm{j}}=1$. Then a mapping of an ATSFWA AOs is: $R^{n}\to R$, and it can be defined as

$$\text{ATSFWA}\ (R_1,R_2,\dots,R_n)=\oplus_{\mathrm{j}=1}^{n}\left(\mathfrak{h}'_{\mathrm{j}}R_{\mathrm{j}}\right)$$

Some fundamental axioms of the ATSFWA AOs are discussed below as follows:

Theorem 1 Consider $R_{\mathrm{j}}=(\rho,\varphi,\tau),(\mathrm{j}=1,2,\dots,n)$ be the family of ATSFNs, and results after aggregation ATSFWA AOs is also TSFN and it can be defined as

$$ATSFWA(R_1,R_2,\dots,R_n)$$
$$=\left(\sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\rho_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\tau_{\mathrm{j}}^{t}\right)^{2}\right)\right)}\right) \tag{5}$$

Proof: For $n=2$,

$$\mathfrak{h}'_1R_1=\left(\sqrt{\mathscr{y}^{-1}\left(\mathfrak{h}'_1\mathscr{y}((\rho_1^{t})^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\varphi_1^{t})^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\tau_1^{t})^2)\right)}\right)$$

And

$$\mathfrak{h}'_2R_2=\left(\sqrt{\mathscr{y}^{-1}\left(\mathfrak{h}'_2\mathscr{y}((\rho_2^{t})^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_2\mathfrak{z}((\varphi_2^{t})^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_2\mathfrak{z}((\tau_2^{t})^2)\right)}\right)$$

Now $\mathfrak{h}'_1R_1\oplus\mathfrak{h}'_2R_2=$

$$\left(\begin{array}{c}\sqrt{\mathscr{y}^{-1}\left(\mathscr{y}\left(\mathscr{y}^{-1}\left(\mathfrak{h}'_1\mathscr{y}((\rho_1^{t})^2)\right)\right)+\mathscr{y}\left(\mathscr{y}^{-1}\left(\mathfrak{h}'_2\mathscr{y}((\rho_2^{t})^2)\right)\right)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\varphi_1^{t})^2)\right)\right)+\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}'_2\mathfrak{z}((\varphi_2^{t})^2)\right)\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\tau_1^{t})^2)\right)\right)+\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}'_2\mathfrak{z}((\tau_2^{t})^2)\right)\right)\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt{\mathscr{y}^{-1}\left(\mathfrak{h}'_1\mathscr{y}((\rho_1^{t})^2)+\mathfrak{h}'_2\mathscr{y}((\rho_2^{t})^2)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\varphi_1^{t})^2)+\mathfrak{h}'_2\mathfrak{z}((\varphi_2^{t})^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}'_1\mathfrak{z}((\tau_1^{t})^2)+\mathfrak{h}'_2\mathfrak{z}((\tau_2^{t})^2)\right)}\end{array}\right)$$

$$=\left(\sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{2}\mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\rho_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{2}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{2}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\tau_{\mathrm{j}}^{t}\right)^{2}\right)\right)}\right)$$

Thus, the statement is true for $n=2$. Now consider the statement is true for $n=k$.

$$ATSFWA(R_1,R_2,\dots,R_n)$$
$$=\left(\sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{k}\mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\rho_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{k}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^{2}\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{k}\mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\tau_{\mathrm{j}}^{t}\right)^{2}\right)\right)}\right)$$

Now taking $n=k+1$, then we have

$$ATSFWA(R_1,R_2,\dots,R_k,R_{k+1})=ATSFWA(R_1,R_2,\dots,R_k)\oplus\mathfrak{h}'_{k+1}R_{k+1}$$

$$=\left(\sqrt{\mathcal{y}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathcal{y}\left(\left(\rho_j^t\right)^2\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\varphi_j^t\right)^2\right)\right)},\sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\tau_j^t\right)^2\right)\right)}\right)$$

$$\oplus\left(\sqrt{\mathcal{y}^{-1}\left(\mathfrak{h}_{k+1}'\mathcal{y}((\rho_{k+1}^t)^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}_{k+1}'\mathfrak{z}((\varphi_{k+1}^t)^2)\right)},\sqrt{\mathfrak{z}^{-1}\left(\mathfrak{h}_{k+1}'\mathfrak{z}((\tau_{k+1}^t)^2)\right)}\right)$$

$$=\left(\begin{array}{c}\sqrt{\mathcal{y}^{-1}\left(\mathcal{y}\left(\mathcal{y}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathcal{y}\left(\left(\rho_j^t\right)^2\right)\right)\right)+\mathcal{y}\left(\mathcal{y}^{-1}\left(\mathfrak{h}_{k+1}'\mathcal{y}((\rho_{k+1}^t)^2)\right)\right)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\varphi_j^t\right)^2\right)\right)\right)+\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}_{k+1}'\mathfrak{z}((\varphi_{k+1}^t)^2)\right)\right)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\tau_j^t\right)^2\right)\right)\right)+\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{h}_{k+1}'\mathfrak{z}((\tau_{k+1}^t)^2)\right)\right)\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt{\mathcal{y}^{-1}\left(\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathcal{y}\left(\left(\rho_j^t\right)^2\right)+\mathfrak{h}_{k+1}'\mathcal{y}((\rho_{k+1}^t)^2)\right)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\varphi_j^t\right)^2\right)+\mathfrak{h}_{k+1}'\mathfrak{z}((\varphi_{k+1}^t)^2)\right)\right)},\\ \sqrt{\mathfrak{z}^{-1}\left(\left(\sum_{j=1}^{k}\mathfrak{h}_j'\mathfrak{z}\left(\left(\tau_j^t\right)^2\right)+\mathfrak{h}_{k+1}'\mathfrak{z}((\tau_{k+1}^t)^2)\right)\right)}\end{array}\right)$$

Hence, the statement is true for $n = k + 1$.

- **(Algebraic)** If $\mathfrak{z}(r) = -\log r$, then ATSFWA operator can be reduced into the TSF weighted averaging TSFWA operator, which can be defined as given below:

$$TSFWA(R_1, R_2, \dots, R_n) = \sqrt{1-\prod_{j=1}^{n}\left(1-\rho_j^t\right)^{\mathfrak{h}_j'},\prod_{j=1}^{n}\left(\varphi_j^t\right)^{\mathfrak{h}_j'},\prod_{j=1}^{n}\left(\tau_j^t\right)^{\mathfrak{h}_j'}} \quad (6)$$

- **(Einstein)** If $\mathfrak{z}(r) = -\log\left((2-r)/r\right)$, then ATSFEWA operator reduces in the TSFEWA operator, which can be defined as given below

$$TSFEWA(R_1, R_2, \dots, R_n) = \left(\begin{array}{c} \dfrac{\sqrt{\prod_{j=1}^{n}\left(1+\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'} - \prod_{j=1}^{n}\left(1-\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'}}}{\sqrt{\prod_{j=1}^{n}\left(1+\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'} + \prod_{j=1}^{n}\left(1-\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'}}}, \\ \dfrac{\sqrt{2}\prod_{j=1}^{n}\left(\varphi_j^t\right)^{\mathfrak{h}_j'}}{\sqrt{\prod_{j=1}^{n}\left(2-\left(\varphi_j^t\right)^2\right)^{\mathfrak{h}_j'} + \prod_{j=1}^{n}\left(\left(\varphi_j^t\right)^2\right)^{\mathfrak{h}_j'}}}, \\ \dfrac{\sqrt{2}\prod_{j=1}^{n}\left(\tau_j^t\right)^{\mathfrak{h}_j'}}{\sqrt{\prod_{j=1}^{n}\left(2-\left(\tau_j^t\right)^2\right)^{\mathfrak{h}_j'} + \prod_{j=1}^{n}\left(\left(\tau_j^t\right)^2\right)^{\mathfrak{h}_j'}}} \end{array} \right) \tag{7}$$

- **(Hamacher)** If $\mathfrak{z}(r) = -\log\left(\left(\theta + (1-\theta)\right)/r\right), \theta > 0$, then the ATSFWA operator reduces into the TSFHWA operator defined as

$$TSFHWA(R_1, R_2, \dots, R_n) = \left(\begin{array}{c} \dfrac{\sqrt{\prod_{j=1}^{n}\left(1+(\theta-1)\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'} - \prod_{j=1}^{n}\left(1-\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'}}}{\sqrt{\prod_{j=1}^{n}\left(1+(\theta-1)\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'} + (\theta-1)\prod_{j=1}^{n}\left(1-\left(\rho_j^t\right)^2\right)^{\mathfrak{h}_j'}}}, \\ \dfrac{\sqrt{\theta}\prod_{j=1}^{n}\left(\varphi_j^t\right)^{\mathfrak{h}_j'}}{\sqrt{\prod_{j=1}^{n}\left(1+(\theta-1)\left(1-\left(\varphi_j^t\right)^2\right)\right)^{\mathfrak{h}_j'} + (\theta-1)\prod_{j=1}^{n}\left(\left(\varphi_j^t\right)^2\right)^{\mathfrak{h}_j'}}}, \\ \dfrac{\sqrt{\theta}\prod_{j=1}^{n}\left(\tau_j^t\right)^{\mathfrak{h}_j'}}{\sqrt{\prod_{j=1}^{n}\left(1+(\theta-1)\left(1-\left(\tau_j^t\right)^2\right)\right)^{\mathfrak{h}_j'} + (\theta-1)\prod_{j=1}^{n}\left(\left(\tau_j^t\right)^2\right)^{\mathfrak{h}_j'}}} \end{array} \right) \tag{8}$$

If we take θ is 1 and 2 in this equation (9), then the TSFHWA operator turns into the TSFWA and TSFEWA operators, respectively.

- **(Frank)** If $\mathfrak{z}(r) = -\log\left(\frac{(\theta-1)}{(\theta^r-1)}\right), \theta > 1$, then the ATSFFWA operator turns into the TSFFWA operator defined as

$$TSFFWA(R_1, R_2, \dots, R_n) = \left(\begin{array}{c} \sqrt{1 - \dfrac{\log\left(1+\prod_{j=1}^{n}\left(\theta^{1-\left(\rho_j^t\right)^2}-1\right)^{\mathfrak{h}_j'}\right)}{\log\theta}}, \\ \sqrt{\dfrac{\log\left(1+\prod_{j=1}^{n}\left(\theta^{\left(\varphi_j^t\right)^2}-1\right)^{\mathfrak{h}_j'}\right)}{\log\theta}}, \\ \sqrt{\dfrac{\log\left(1+\prod_{j=1}^{n}\left(\theta^{\left(\tau_j^t\right)^2}-1\right)^{\mathfrak{h}_j'}\right)}{\log\theta}} \end{array} \right) \tag{10}$$

Theorem 2 (Boundedness). Let the family TSFNs $R_j = \left(\rho_j^t, \varphi_j^t, \tau_j^t\right)$ where $(j = 1, 2, \dots, n)$ and consider

$$\rho_{mjn} = \min\left(\rho_j^t\right)_{min} \ \forall j = 1, 2, \dots, n$$
$$\rho_{max} = \max\left(\rho_j^t\right)_{min} \ \forall j = 1, 2, \dots, n$$
$$\varphi_{min} = \min\left(\varphi_j^t\right)_{min} \ \forall j = 1, 2, \dots, n$$

$$\varphi_{max} = \max(\varphi_{j}^{t})_{max}\ \forall j = 1, 2, \dots, n$$
$$\tau_{min} = \min(\tau_{j}^{t})_{min}\ \forall j = 1, 2, \dots, n$$
$$\tau_{max} = \max(\tau_{j}^{t})_{max}\ \forall j = 1, 2, \dots, n$$

Let $R^{+} = \left(\rho_{j_{min}}^{t}, \varphi_{j_{max}}^{t}, \tau_{j_{max}}^{t}\right)$ and $R^{+} = \left(\rho_{j_{max}}^{t}, \varphi_{j_{min}}^{t}, \tau_{j_{min}}^{t}\right)$

$R^{+} \leq ATSFWA(R_1, R_2, \dots, R_n) \leq R^{-}$ (8)

Let $ATSFWA(R_1, R_2, \dots, R_n) = R$

$$= \left(\sqrt{\mathcal{y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathcal{y}\left(\rho_j^t\right)^2\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathfrak{z}\left(\varphi_j^t\right)^2\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathfrak{z}\left(\tau_j^t\right)^2\right)}\right)$$

Proof: Consider for any $(i = 1, 2, \dots, n)$ we have $\rho_{i_{min}}^{t} \leq \rho_{i}^{t} \leq \rho_{i_{max}}^{t}$

$$\left(\rho_{i_{min}}^{t}\right)^2 \leq (\rho_i^t)^2 \leq \left(\rho_{i_{max}}^{t}\right)^2$$

Since $\mathcal{y}(r), (\mathcal{y})\epsilon[0, 1]$ be the monotonically strictly increasing function of $\mathcal{y}(r)$, so we have

$$\mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}\left(\rho_{i_{min}}^{t}\right)^2\right) \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\rho_i^t)^2\right) \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} w_i' \mathcal{y}\left(\rho_{i_{max}}^{t}\right)^2\right)$$

$$\Rightarrow \left(\rho_{i_{min}}^{t}\right)^2 \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\rho_i^t)^2\right) \leq \left(\rho_{i_{max}}^{t}\right)^2 \quad (11)$$

Now for any $(i = 1, 2, \dots, n)$, we have $\left(\varphi_{i_{min}}^{t}\right)^2 \leq (\varphi_i^t)^2 \leq \left(\varphi_{i_{max}}^{t}\right)^2$ since $\mathfrak{z}(r), [0, 1]$ be the decreasing function. Then we get

$$\mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}\left(\varphi_{i_{min}}^{t}\right)^2\right) \leq \mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\varphi_i^t)^2\right) \leq \mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}\left(\varphi_{i_{max}}^{t}\right)^2\right)$$

$$\Rightarrow \left(\varphi_{i_{min}}^{t}\right)^2 \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\varphi_i^t)^2\right) \leq \left(\varphi_{i_{max}}^{t}\right)^2 \quad (12)$$

Any number $(i = 1, 2, \dots, n)$, we have $\left(\tau_{i_{min}}^{t}\right)^2 \leq (\tau_i^t)^2 \leq \left(\tau_{i_{max}}^{t}\right)^2$ and $\mathfrak{z}(r) \in [0, 1]$ be the strictly decreasing function of $\mathfrak{z}(r)$. Then we get

$$\mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}\left(\tau_{i_{min}}^{t}\right)^2\right) \leq \mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\tau_i^t)^2\right) \leq \mathfrak{z}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}\left(\tau_{i_{max}}^{t}\right)^2\right)$$

$$\Rightarrow \left(\tau_{i_{min}}^{t}\right)^2 \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\tau_i^t)^2\right) \leq \left(\tau_{i_{max}}^{t}\right)^2 \quad (13)$$

From equations $10 - 12$, We can observe that

$$\left(\rho_{i_{min}}^{t}\right)^2 - \left(\varphi_{i_{max}}^{t}\right)^2 - \left(\tau_{i_{max}}^{t}\right)^2 \leq \mathcal{y}^{-1}\left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\rho_i^t)^2\right)\mathcal{y}^{-1} - \left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\varphi_i^t)^2\right)\mathcal{y}^{-1} - \left(\sum_{i=1}^{n} \mathfrak{h}_i' \mathcal{y}(\tau_i^t)^2\right)$$
$$\leq \left(\rho_{i_{min}}^{t}\right)^2 - \left(\varphi_{i_{max}}^{t}\right)^2 \leq \left(\tau_{i_{max}}^{t}\right)^2$$
$$S(R^{+}) \leq S\left(ATSFWA(R_1, R_2, \dots, R_n)\right) \leq S(R^{-})$$

Therefore $R^{+} \leq ATSFWA(R_1, R_2, \dots, R_n) \leq R^{-}$

Theorem 3 (Monotonicity) Consider $R_j (j = 1, 2, \dots, n)$ be the family of TSFNs, $\mathfrak{h}_i' \epsilon [0, 1]$ $(j = 1, 2, \dots, n)$ be their WV and fulfill the condition $\sum_{j=1}^{n} \mathfrak{h}_i' = 1$, if R be the TSFN, then

$$ATSFWA(R_1 \oplus R, R_2 \oplus R, \dots, R_n \oplus R) = ATSFWA(R_1, R_2, \dots, R_n) \oplus R$$

Proof:

$$R_j \oplus R = \left(\sqrt{\mathcal{y}^{-1}\left(\mathcal{y}((\rho_i^t)^2) + \mathcal{y}((\rho^t)^2)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}((\varphi_i^t)^2) + \mathfrak{z}((\varphi^t)^2)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}((\tau_i^t)^2) + \mathfrak{z}((\tau^t)^2)\right)}\right)$$

Let

$$ATSFWA(R_1 \oplus R, R_2 \oplus R, \dots, R_n \oplus R) = \begin{pmatrix} \sqrt{\mathscr{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathscr{Y}\left(\mathscr{Y}^{-1}\left(\mathscr{Y}((\rho_i^t)^2) + \mathscr{Y}((\rho^t)^2)\right)\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{z}((\varphi_i^t)^2) + \mathfrak{z}((\varphi^t)^2)\right)\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}\left(\mathfrak{z}^{-1}\left(\mathfrak{z}((\tau_i^t)^2) + \mathfrak{z}((\tau^t)^2)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\mathscr{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathscr{Y}((\rho_i^t)^2) + \mathscr{Y}((\rho^t)^2)\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathfrak{z}((\varphi_i^t)^2) + \mathfrak{z}((\varphi^t)^2)\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathfrak{z}((\tau_i^t)^2) + \mathfrak{z}((\tau^t)^2)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\mathscr{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathscr{Y}((\rho_i^t)^2) + \mathscr{Y}((\rho^t)^2)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}((\varphi_i^t)^2) + \mathfrak{z}((\varphi^t)^2)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}((\tau_i^t)^2) + \mathfrak{z}((\tau^t)^2)\right)} \end{pmatrix}$$

Now

$$ATSFWA(R_1, R_2, \dots, R_n) \oplus R$$

$$= \left(\sqrt{\mathscr{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathscr{Y}((\rho_i^t)^2)\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}((\varphi_i^t)^2)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \mathfrak{z}((\tau_i^t)^2)\right)}\right) \oplus (\rho^t, \varphi^t, \tau^t)$$

$$= \begin{pmatrix} \sqrt{\mathscr{Y}^{-1}\left(\mathscr{Y}\left(\mathscr{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathscr{Y}((\rho_i^t)^2)\right)\right) + \mathscr{Y}(\rho^t)^2\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathfrak{z}((\varphi_i^t)^2)\right)\right) + \mathfrak{z}(\varphi^t)^2\right)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\mathfrak{z}\left(\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i' \left(\mathfrak{z}((\tau_i^t)^2)\right)\right) + \mathfrak{z}(\tau^t)^2\right)\right)} \end{pmatrix}$$

$$= \left(\begin{array}{c} \sqrt{\mathcal{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'(\mathcal{Y}((\rho_i^t)^2) + \mathcal{Y}(\rho^t)^2)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'(\mathfrak{z}((\varphi_i^t)^2) + \mathfrak{z}(\varphi^t)^2)\right)}, \\ \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'(\mathfrak{z}((\tau_i^t)^2) + \mathfrak{z}(\tau^t)^2)\right)} \end{array} \right)$$

Hence proved the theorem.

Theorem 4 (Idempotency) Consider if all $R_j = (\rho, \varphi, \tau)$ $(j = 1, 2, \dots, n)$ are equal and consider $R_j = R = (\rho, \varphi, \tau)$ $\forall$ $(j = 1, 2, \dots, n)$, then

$$ATSFWA(R_1, R_2, \dots, R_n) = R$$

Proof:

$$ATSFWA(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathcal{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'\left(\mathcal{Y}((\rho_i^t)^2)\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'\left(\mathfrak{z}((\varphi_i^t)^2)\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_i'\left(\mathfrak{z}((\tau_i^t)^2)\right)\right)} \right)$$

Since $R_j = (\rho_i, \varphi_i, \tau_i) = (\varphi^t, \varphi^t, \tau^t)$ $\forall$ $(j = 1, 2, \dots, n)$ then we have

$$ATSFWA(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathcal{Y}^{-1}\left(\left(\mathcal{Y}((\rho_i^t)^2)\right)\sum_{j=1}^{n} \mathfrak{h}_i'\right)}, \sqrt{\mathfrak{z}^{-1}\left(\left(\mathfrak{z}((\varphi_i^t)^2)\right)\sum_{j=1}^{n} \mathfrak{h}_i'\right)}, \sqrt{\mathfrak{z}^{-1}\left(\left(\mathfrak{z}((\tau_i^t)^2)\right)\sum_{j=1}^{n} \mathfrak{h}_i'\right)} \right)$$

$$= (\rho, \varphi, \tau) = R$$

Hence the theorem.

Definition 14 Consider $R_j = (\rho, \varphi, \tau)$ be any family of TSFNs. Then TSFOWA operator of dimension n is mapping ATSFOWA: $R^n \to R^*$ is defined as

$$ATSFOWA(\rho, \varphi, \tau) = \bigoplus_{i=1}^{n} \left(\mathfrak{h}_i' R_{\sigma(i)}\right)$$

Then ATSFWA is said to be the ATSFOWA operator, whereas WV $\mathfrak{h}_i' = (\mathfrak{h}_1', \mathfrak{h}_2', \dots, \mathfrak{h}_n')$ and $\sum_{j=1}^{n} \mathfrak{h}_i' = 1$ and σ said to be the permutation with the condition $\sigma(i-1) \geq \sigma(i)$ $\forall$ $(i = 1, 2, \dots, n)$. By utilizing TSF operational laws on TSFNs, we demonstrate the following theorem.

Theorem 5 Consider $R_j = (\rho, \varphi, \tau)$ be any family of TSFNs. Then the aggregation results of the TSFOWA operator are also TSFNs given by

$$ATSFWA(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathcal{Y}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathcal{Y}\left(\left(\rho_{\sigma(j)}^t\right)^2\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathfrak{z}\left(\left(\varphi_{\sigma(j)}^t\right)^2\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{j=1}^{n} \mathfrak{h}_j' \mathfrak{z}\left(\left(\tau_{\sigma(j)}^t\right)^2\right)\right)} \right) \quad (14)$$

Theorem 6 (Idempotency) Consider the family of TSFNs $R_j = (\rho, \varphi, \tau)$ is defined as $R_j = R$ and $\mathfrak{h}_i'$ be the WV. Then $ATSFOWA(R_1, R_2, \dots, R_n) = R$

Theorem 7 ATSFWA (Boundedness) Consider the family of TSFNs $R_j = (\rho, \varphi, \tau)$ is and $\mathfrak{h}_i'$ be the WV. Consider $R^- = \min(R_j)$ and $R^+ = \max(R_j)$. Then $ATSFOWA\ R^- \leq (R_1, R_2, \dots, R_n) = R^+$.

Theorem 8 (Monotonicity) Consider $R_j = (\rho, \varphi, \tau), (j = 1, 2, \dots, n)$ and $R_j^t = (\rho^t, \varphi^t, \tau^t)$ be any two sets of TSFNs and $\mathfrak{h}_j'$ be the WV. And if $R_j \leq R_j^t$ $\forall$ j. $ATSFOWA\ R_j\ (\rho, \varphi, \tau) \leq ATSFOWA\ (\rho^t, \varphi^t, \tau^t)$.

Definitions 13 and 14 make it clear that the ATSFWA and ATSFOWA operators aggregate TSFNs by only weighting them and by ordering their weighting, respectively. As a result, weights demonstrate the many aspects of both ATSFWA and ATSFOWA operators. This shortcoming is not covered by any of the operators. To solve the problem, we define the ATSFHWA operator as follows.

Definition 15 Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be any set of TSFNs and $\mathfrak{h}'_{\mathrm{i}}$ be the WV. Then TSFHWA operator of dimension n is mapping ATSFHWA: $R^n \to R^*$ can be represented as

$$ATSFHWA(\rho, \varphi, \tau) = \bigoplus_{\mathrm{j}=1}^{n} (\mathfrak{h}'_{\mathrm{i}})\ddot{R}_{\sigma(\mathrm{j})}$$

Where $\ddot{R}_{\sigma(\mathrm{j})} = cR_{\mathrm{j}}$ and c is any balancing coefficient and $\left(\ddot{R}_{\sigma(1)}, \ddot{R}_{\sigma(2)}, \dots, \ddot{R}_{\sigma(n)}\right)$ is the permutation of j for weights TSFNs.

Almost all characteristics of the ATSFHWA operator are similar to ATSFWA that we discussed in detail in Theorem 1, 2, 3, and 4. However, as stated in the following Theorem 8, the ATSFHWA tool is a stronger model of the ATSFOWA operator.

Theorem 9 The ATSFHWA operator has special cases known as the ATSFWA and ATSFOWA operators.

Proof: Consider $\mathfrak{h}'_{\mathrm{i}} = \left(\frac{1}{\mathfrak{h}'_1}, \frac{1}{\mathfrak{h}'_2}, \dots, \frac{1}{\mathfrak{h}'_{\mathrm{n}}}\right)$, by Definition 15, we have ATSFHWA $(R_1, R_2, \dots, R_n) = \left(\mathfrak{h}'_1 \oplus \ddot{R}_{\sigma(1)} \oplus \mathfrak{h}'_2\ddot{R}_{\sigma(2)} \oplus \dots \oplus \mathfrak{h}'_{\mathrm{n}}\ddot{R}_{\sigma(n)}\right) = \frac{1}{c}\left(\mathfrak{h}'_1 \oplus \ddot{R}_{\sigma(1)} \oplus \mathfrak{h}'_2\ddot{R}_{\sigma(2)} \oplus \dots \oplus \mathfrak{h}'_{\mathrm{n}}\ddot{R}_{\sigma(n)}\right) = (\mathfrak{h}'_1R_1 \oplus \mathfrak{h}'_2R_2 \oplus \dots \oplus \mathfrak{h}'_{\mathrm{n}}R_n) = ATSFWA(R_1, R_2, \dots, R_n)$. Similarly, we can also explain that ATSFOWA is a unique case of ATSFHWA.

5. T-Spherical Fuzzy Archimedean Geometric Averaging AOs

In this part, we proposed ATSFWG, ATSFOWG, and ATSFHWG operators based on Archimedean operational laws and discuss their fundamental characteristics.

Definition 16 Consider $R_{\mathrm{j}}(1, 2, \dots, n)$ be the family of TSFNs, and $\mathfrak{h}'_1 = (\mathfrak{h}'_1, \mathfrak{h}'_2, \dots, \mathfrak{h}'_{\mathrm{n}})^t$ be the WV with $\mathfrak{h}'_{\mathrm{i}} \epsilon [0, 1]$ and $\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{i}} = 1$. Then ATCN and ATN are based on the TSF geometric (TSFWG) operator with mapping $R^n \to R^*$.

$$ATSFWG(R_1, R_2, \dots, R_n) = \bigotimes_{\mathrm{j}=1}^{n} \left(R_{\mathrm{j}}^{\mathfrak{h}'_{\mathrm{i}}}\right)$$

Theorem 10 Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau), (\mathrm{j} = 1, 2, \dots, n)$ be the set of ATSFNs, and then the aggregation outcomes by using the ATSFWG AOs is also in the form of TSFN and it can be defined as:

$$ATSFWG(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\rho_{\mathrm{j}}^t\right)^2\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\varphi_{\mathrm{j}}^t\right)^2\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\tau_{\mathrm{j}}^t\right)^2\right)\right)}\right) \quad (15)$$

Proof: The Proof of this theorem is the same as like Theorem 1.

Definition 17 Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be any family of TSFNs. Then TSFOWG operator of dimension n is mapping on ATSFOWG: $R^n \to R^*$ is defined as

$$ATSFOWG(\rho, \varphi, \tau) = \bigotimes_{\mathrm{j}=1}^{n} \left(\mathfrak{h}'_{\mathrm{i}}R_{\sigma(\mathrm{j})}\right)$$

Then ATSFWG is said to be the ATSFOWG operator, whereas WV $\mathfrak{h}'_{\mathrm{i}} = (\mathfrak{h}'_1, \mathfrak{h}'_2, \dots, \mathfrak{h}'_{\mathrm{n}})$ and $\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{i}} = 1$ and σ is said to be the permutation with the condition $\sigma(\mathrm{j}-1) \geq \sigma(\mathrm{j})\ \forall\ (\mathrm{j} = 1, 2, \dots, n)$. By utilizing TSF operational laws on TSFNs, we demonstrate the following theorem.

Theorem 5 Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be any family of TSFNs. Then the aggregation results of the TSFOWG operator are also TSFNs given by

$$ATSFWG(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathfrak{z}\left(\left(\rho_{\sigma(\mathrm{j})}^t\right)^2\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\varphi_{\sigma(\mathrm{j})}^t\right)^2\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n} \mathfrak{h}'_{\mathrm{j}}\mathscr{y}\left(\left(\tau_{\sigma(\mathrm{j})}^t\right)^2\right)\right)}\right) \quad (16)$$

Theorem 11 (Idempotency) Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be the family of TSFNs is defined as $R_{\mathrm{j}} = R$ and $\mathfrak{h}'_{\mathrm{i}}$ be the WV. Then $ATSFOWG(R_1, R_2, \dots, R_n) = R$

Theorem 12 ATSFWG (Boundedness) Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be the family of TSFNs and $\mathfrak{h}'_{\mathrm{i}}$ be the WV. Consider $R^- = \min(R_{\mathrm{j}})$ and $R^+ = \max(R_{\mathrm{j}})$. Then $ATSFOWG\ R^- \leq (R_1, R_2, \dots, R_n) = R^+$.

Theorem 13 (Monotonicity) Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau), (\mathrm{j} = 1, 2, \dots, n)$ and $R_{\mathrm{j}}^t = (\rho^{\mathrm{t}}, \varphi^{\mathrm{t}}, \tau^{\mathrm{t}})$ be any two sets of TSFNs and $\mathfrak{h}'_{\mathrm{j}}$ be the WV. And if $R_{\mathrm{j}} \leq R_{\mathrm{j}}^t\ \forall\ \mathrm{j}$. $ATSFOWG\ R_{\mathrm{j}}\ (\rho, \varphi, \tau) \leq ATSFOWA\ (\rho^{\mathrm{t}}, \varphi^{\mathrm{t}}, \tau^{\mathrm{t}})$.

Definition 18 Consider $R_{\mathrm{j}} = (\rho, \varphi, \tau)$ be any set of TSFNs and $\mathfrak{h}'_{\mathrm{i}}$ be the WV. Then TSFHWG operator of dimension n is mapping ATSFHWG: $R^n \to R^*$ can be written as

$$ATSFHWG(\rho, \varphi, \tau) = \bigotimes_{\mathrm{j}=1}^{n} (\mathfrak{h}'_{\mathrm{i}})\ddot{R}_{\sigma(\mathrm{j})}$$

Where $\ddot{R}_{\sigma(\mathrm{j})} = cR_{\mathrm{j}}$ and c is any balancing coefficient and $\left(\ddot{R}_{\sigma(1)}, \ddot{R}_{\sigma(2)}, \dots, \ddot{R}_{\sigma(n)}\right)$ is the permutation j with the weight of TSFNs.
Almost all characteristics of the ATSFHWG operator are similar to ATSFWG which we discussed in detail in Theorem 1, 2, 3, and 4. However, as stated in the following Theorem 8, the ATSFHWG operator is a stronger model of the ATSFOWG operator.

Theorem 14 The ATSFHWG operator has special cases known as the ATSFWG and ATSFOWG operators.

Proof: Consider $\mathfrak{h}'_{\mathrm{i}} = \left(\frac{1}{\mathfrak{h}'_1}, \frac{1}{\mathfrak{h}'_2}, \dots, \frac{1}{\mathfrak{h}'_{\mathrm{n}}}\right)$, by Definition 15, we have ATSFHWG $(R_1, R_2, \dots, R_n) = \left(\mathfrak{h}'_1 \otimes \ddot{R}_{\sigma(1)} \otimes \mathfrak{h}'_2 \ddot{R}_{\sigma(2)} \otimes \dots \otimes \mathfrak{h}'_{\mathrm{n}} \ddot{R}_{\sigma(n)}\right) = \frac{1}{c}\left(\mathfrak{h}'_1 \otimes \ddot{R}_{\sigma(1)} \otimes \mathfrak{h}'_2 \ddot{R}_{\sigma(2)} \otimes \dots \otimes \mathfrak{h}'_{\mathrm{n}} \ddot{R}_{\sigma(n)}\right) = (\mathfrak{h}'_1 R_1 \otimes \mathfrak{h}'_2 R_2 \otimes \dots \otimes \mathfrak{h}'_{\mathrm{n}} R_n) = ATSFWG(R_1, R_2, \dots, R_n)$. Similarly, we can also explain that ATSFOWG is a unique case of ATSFHWG.

Remark The following changes can be observed in the weighted geometric operator as given below:

- **(Algebraic)** If $\mathfrak{z}(r) = -\log r$, then ATSFWG operator can be reduced into the TSF weighted averaging TSFWG operator, which can be defined as given below:

$$TSFWG(R_1, R_2, \dots, R_n) = \prod_{\mathrm{j}=1}^{n}\left(\tau_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}, \prod_{\mathrm{j}=1}^{n}\left(\varphi_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}, \sqrt{1 - \prod_{\mathrm{j}=1}^{n}\left(1 - \tau_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}} \tag{17}$$

- **(Einstein)** If $\mathfrak{z}(r) = -\log\left((2 - r)/r\right)$, then ATSFEWA operator reduces in the TSFEWA operator, which can be defined as given below

$$TSFEWG(R_1, R_2, \dots, R_n) = \left(\begin{array}{c} \dfrac{\sqrt{2}\prod_{\mathrm{j}=1}^{n}\left(\rho_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(2 - \left(\rho_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{n}\left(\left(\rho_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}, \\ \dfrac{\sqrt{2}\prod_{\mathrm{j}=1}^{n}\left(\varphi_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(2 - \left(\varphi_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{n}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}, \\ \dfrac{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{n}\left(1 - \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{n}\left(1 - \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}} \end{array}\right) \tag{18}$$

- **(Hamacher)** If $\mathfrak{z}(r) = -\log\left(\left(\theta + (1 - \theta)\right)/r\right), \theta > 0$, then the ATSFWA operator reduces into the TSFHWA operator defined as

$$TSFHWG(R_1, R_2, \dots, R_n) = \left(\begin{array}{c} \dfrac{\sqrt{\theta}\prod_{\mathrm{j}=1}^{n}\left(\rho_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + (\theta - 1)\left(1 - \left(\rho_{\mathrm{j}}^{t}\right)^2\right)\right)^{\mathfrak{h}'_{\mathrm{j}}} + (\theta - 1)\prod_{\mathrm{j}=1}^{n}\left(\left(\rho_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}, \\ \dfrac{\sqrt{\theta}\prod_{\mathrm{j}=1}^{n}\left(\varphi_{\mathrm{j}}^{t}\right)^{\mathfrak{h}'_{\mathrm{j}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + (\theta - 1)\left(1 - \left(\varphi_{\mathrm{j}}^{t}\right)^2\right)\right)^{\mathfrak{h}'_{\mathrm{j}}} + (\theta - 1)\prod_{\mathrm{j}=1}^{n}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}, \\ \dfrac{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + (\theta - 1)\left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{n}\left(1 - \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}}{\sqrt{\prod_{\mathrm{j}=1}^{n}\left(1 + (\theta - 1)\left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}} + (\theta - 1)\prod_{\mathrm{j}=1}^{n}\left(1 - \left(\tau_{\mathrm{j}}^{t}\right)^2\right)^{\mathfrak{h}'_{\mathrm{j}}}}} \end{array}\right) \tag{19}$$

If we take θ is 1 and 2 in this equation (19), then the TSFHWA operator turns into the TSFWA and TSFEWA operators, respectively.

- **(Frank)** If $\mathfrak{z}(r) = -\log\left(\frac{(\theta - 1)}{(\theta^{r} - 1)}\right), \theta > 1$, then the ATSFFWA operator turns into the TSFFWA operator defined as

$$TSFFWG(R_1, R_2, \dots, R_n) = \begin{pmatrix} \sqrt{\frac{\log\left(1+\prod_{\mathrm{j}=1}^{n}\left(\theta^{\left(\rho_{\mathrm{j}}^{t}\right)^{2}}-1\right)^{\mathfrak{h}_{\mathrm{j}}'}\right)}{\log\theta}}, \\ \sqrt{\frac{\log\left(1+\prod_{\mathrm{j}=1}^{n}\left(\theta^{\left(\varphi_{\mathrm{j}}^{t}\right)^{2}}-1\right)^{\mathfrak{h}_{\mathrm{j}}'}\right)}{\log\theta}}, \\ \sqrt{1-\frac{\log\left(1+\prod_{\mathrm{j}=1}^{n}\left(\theta^{1-\left(\tau_{\mathrm{j}}^{t}\right)^{2}}-1\right)^{\mathfrak{h}_{\mathrm{j}}'}\right)}{\log\theta}} \end{pmatrix} \quad (20)$$

6. Propose algorithm for solving MADM problem

In this section, under the TSF environment, we solve the MADM problem by utilizing the proposed ATSFWA and ATSFWG operators. For this, we are taking $F = \{f_1, f_2, \dots, f_n\}$ be the family of alternatives, $G = \{g_1, g_2, \dots, g_n\}$ be the family of attributes, and $\mathfrak{h}_{\mathrm{i}}' = \{\mathfrak{h}_1', \mathfrak{h}_2', \dots, \mathfrak{h}_{\mathrm{n}}'\}$ be the WV with condition $\mathfrak{h}_{\mathrm{i}}' \epsilon [0, 1]$, and $\sum_{\mathrm{j}=1}^{n} \mathfrak{h}_{\mathrm{i}}' = 1$. Consider the TSF decision matrix can be represented as $R = \left(r_{\mathrm{i}j}\right)_{m\times n}$. Then the proposed ATSFWA and ATSFWG operators are applied to solve the MADM problem for the TSF system. The proposed algorithm is explained by using the following steps:

Step 1. Firstly, aggregate the TSFNs in the decision matrix $R_{\mathrm{i}j}$, for each alternative G, by utilizing the ATSFWA and ATSFWG operators as follows:

$$ATSFWA(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathscr{y}\left(\left(\rho_{\mathrm{j}}^{t}\right)^{2}\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathfrak{z}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^{2}\right)\right)}, \sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathfrak{z}\left(\left(\tau_{\mathrm{j}}^{t}\right)^{2}\right)\right)}\right)$$

And

$$ATSFWG(R_1, R_2, \dots, R_n) = \left(\sqrt{\mathfrak{z}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathfrak{z}\left(\left(\rho_{\mathrm{j}}^{t}\right)^{2}\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathscr{y}\left(\left(\varphi_{\mathrm{j}}^{t}\right)^{2}\right)\right)}, \sqrt{\mathscr{y}^{-1}\left(\sum_{\mathrm{j}=1}^{n}\mathfrak{h}_{\mathrm{j}}'\mathscr{y}\left(\left(\tau_{\mathrm{j}}^{t}\right)^{2}\right)\right)}\right)$$

Step 2. Calculation of the SV of the aggregated findings by using Definition 2.
Step 3. Evaluate the ranking of alternatives by utilizing Definition 3.
Step 4. End.

7. Numerical Example

In this section, by using the ATSFWA and ATSFWG operators solve the real-world problem. The explanation of the challenging problem is given below:

The health of people has considerable attention in modern society. So, it is necessary that in case of surgery doctors use neat and sterilized instruments. For successful surgery, no doubt the experience of a doctor is very important, but surgical instruments play vital in surgery. Using surgical instruments, doctors can cut through soft tissue, remove bone, dissect and isolate lesions, and eliminate or remove aberrant structures. So, multiple surgical companies offer newly invented and the latest instruments manufactured from machines. So, it is a challenging issue in a time, when many companies offer their best production quality and claim that the functionality of their instruments is very precise and accurate during the surgery. We construct a numerical example for the illustration of our proposed work.

Example 1

Suppose that, a list of four $\widetilde{\daleth}_{\mathrm{l}} = (1, 2, \ldots, 4)$ surgical instruments manufacturing companies. To choose the best company, by the consideration of four attributes $\widetilde{\mathcal{G}}_{\mathrm{l}} = (1, 2, \ldots, 4)$ in our mind, as follows:

i. $\widetilde{\mathcal{G}_1}$ is the nature of the material (stainless steel magnetic or nonmagnetic).
ii. $\widetilde{\mathcal{G}_2}$ purity of the material.
iii. $\widetilde{\mathcal{G}_3}$ is the accuracy in the functionality.
iv. $\widetilde{\mathcal{G}_4}$ is designed by computer numerical control (CNC) machines or by hand.

The WV distributed by the experts is given as $(0.25, 0.25, 0.14, 0.36)^T$. By using the proposed AOs, the decision-maker evaluate the data of four surgical instruments manufacturing companies under the consideration of four attributes $\widetilde{\mathcal{G}}_{\mathrm{l}} = (1, 2, \ldots, 4)$.

Step 1. Collection of fuzzy data by anonymous decision-makers.

Table 1. T-SF decision matrix

$\widetilde{\daleth}_{\mathbf{1}}$	$\widetilde{\daleth}_{\mathbf{2}}$	$\widetilde{\daleth}_{\mathbf{3}}$	$\widetilde{\daleth}_{\mathbf{4}}$	$\widetilde{\daleth}_{\mathbf{5}}$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{1}}}$	$(0.82, 0.61, 0.55)$	$(0.91, 0.35, 0.82)$	$(0.77, 0.51, 0.51)$	$(0.80, 0.71, 0.33)$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{2}}}$	$(0.71, 0.14, 0.32)$	$(0.15, 0.25, 0.33)$	$(0.43, 0.49, 0.31)$	$(0.1, 0.89, 0.44)$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{3}}}$	$(0.85, 0.67, 0.52)$	$(0.16, 0.46, 0.41)$	$(0.13, 0.66)$	$(0.15, 0.45, 0.61)$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{4}}}$	$(0.43, 0.31, 0.46)$	$(0.33, 0.32, 0.69)$	$(0.18, 0.79, 0.77)$	$(0.14, 0.41, 0.71)$

Step 2. The aggregated values by utilizing the ATSFWA and ATSFWG operators are represented in Table 2.

Table 2. Shows Aggregation outcomes

	ATSFWA	ATSFWG
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{1}}}$	$\begin{pmatrix}0.6334,\\0.0997,\\0.1121\end{pmatrix}$	$\begin{pmatrix}0.1126,\\0.5875,\\0.5886\end{pmatrix}$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{2}}}$	$\begin{pmatrix}0.5574,\\0.0678,\\0.0898\end{pmatrix}$	$\begin{pmatrix}0.0438,\\0.6263,\\0.5158\end{pmatrix}$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{3}}}$	$\begin{pmatrix}0.5960,\\0.0889,\\0.0904\end{pmatrix}$	$\begin{pmatrix}0.0448,\\0.5635,\\0.5746\end{pmatrix}$
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{4}}}$	$\begin{pmatrix}0.4881,\\0.0733,\\0.1006\end{pmatrix}$	$\begin{pmatrix}0.0485,\\0.5528,\\0.6023\end{pmatrix}$

Step 3. The aggregation outcomes of the ATSFWA and ATSFWG operators is provided in Table 3.

Table 3. Shows the score value of aggregated data

	ATSFWA	ATSFWG
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{1}}}$	0.1021	-0.1406
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{2}}}$	0.0538	-0.1330
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{3}}}$	0.0752	-0.1195
$\widetilde{\boldsymbol{\mathcal{G}}_{\mathbf{4}}}$	0.0277	-0.1309

Step 4. The ordering of the score values can be seen in Table 4.

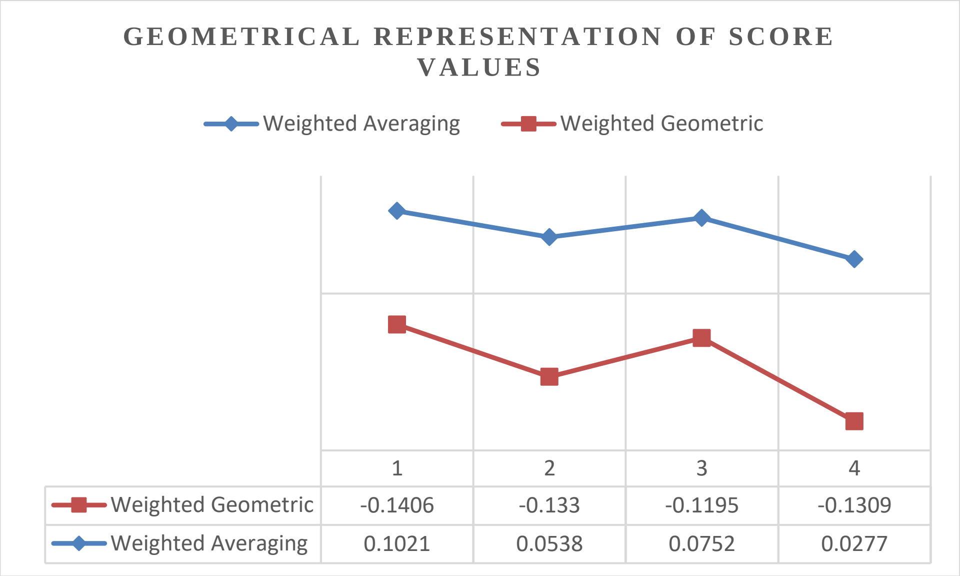

Figure 1. Shows the score function geometrically of Table 3

The blue dots in the graph represents the aggregated values of the ATSFWA operator while the red dots show the aggregated results of the ATSFWG operators.

Step 5. The ranking ordering of the aggregated results by using the proposed AOs is given in Table 4.

Table 4. Ranking of score function

	Ordering
ATSFWA	$\widetilde{\daleth}_1 > \widetilde{\daleth}_3 > \widetilde{\daleth}_2 > \widetilde{\daleth}_4$
ATSFWG	$\widetilde{\daleth}_3 > \widetilde{\daleth}_4 > \widetilde{\daleth}_2 > \widetilde{\daleth}_1$

It is observed in Table 4. by using the ATSFWA operator $\widetilde{\daleth}_1$ is the best option from the list of options, while using the ATSGWG operator $\widetilde{\daleth}_3$ is the best option. It depends upon the experts whether they choose ATSFWA or ATSFWG operator for the aggregation of the data.

8. Comparative Analysis

To express the usefulness and reliability of the constructed idea, we compare the aggregated outcomes of our created AOs to the existing AOs and describe the superiority of ATSFWA and ATSFWG operators. For this, we compare our proposed AOs with TSF Einstein hybrid (TSFEH) weighted averaging (TSFEHWA), TSFEH weighted geometric (TSFEHWG) by Munir et al. (2020), TSF weighted averaging (TSFWA), TSF weighted geometric (TSFWG) by (Ullah et al. 2020a), TSF Hamacher weighted averaging (TSFHWA), TSF Hamacher weighted geometric (TSFHWG) by Ullah et al. (2020), and Dombi TSF Prioritized weighted averaging (DTSFPWA), and Dombi TSF Prioritized weighted geometric (DTSFPWG) by Mahmood et al. (2021). Many other prevailing AOs are unable to aggregate the TSF information due to limitations in their structures for example the idea of Jiang et al. (2018) for IF weighted geometric (IFWG) and IF weighted averaging (IFWA), PyFS weighted averaging (PyFSWA), PyFS weighted geometric (PyFSWG) by (Wei and Lu 2018), and q-ROFS weighted averaging (q-ROFSWA), q-ROFS weighted geometric (q-ROFSWG).

To demonstrate the significance of ATN and ATCN on TSFS theory. Discuss the following important observations by changing the generating function. Also discussed their SF values and rank order in Table 5 as given below:

Table 5. Comparative analysis of proposed AOs

Methods	Operators	Score Values	Ranking Results
Proposed Operators	If $\mathfrak{z}(r) = -\log\big((\theta + (1-\theta))/r\big)$ then ATSFWA turns into TSHWA	$S(c_1) = 0.1021$, $S(c_2) = 0.0538$, $S(c_3) = 0.0752$,	$\daleth_1 > \daleth_3 > \daleth_2 > \daleth_4$

Methods	Operators	Score Values	Ranking Results
		$S(c_4) = 0.0277$	
	If $\mathfrak{z}(r) = -\log\left(\left(\theta + (1-\theta)\right)/r\right)$ then ATSFWG turns into TSHWG	$S(c_1) = -0.1406$, $S(c_2) = -0.1330$, $S(c_3) = -0.1195$, $S(c_4) = -0.1309$	$\daleth_3 > \daleth_4 > \daleth_2 > \daleth_1$
Ullah et al. (2020a)	If $\mathfrak{z}(r) = -\log r$ then ATSFWA turns into TSWA	$S(c_1) = 0.2543$, $S(c_2) = 0.0056$, $S(c_3) = 0.0285$, $S(c_4) = 0.0009$	$\daleth_1 > \daleth_3 > \daleth_2 > \daleth_4$
	If $\mathfrak{z}(r) = -\log r$ then ATSFWA turns into TSWA	$S(c_1) = 0.0773$, $S(c_2) = 0.004$, $S(c_3) = 0.0107$, $S(c_4) = 0.0461$	$\daleth_1 > \daleth_4 > \daleth_3 > \daleth_2$
Munir et al. (2020)	If $\mathfrak{z}(r) = -\log\left((2-r)/r\right)$ then ATSFEWA turns into TSFEWA	$S(c_1) = 0.0636$, $S(c_2) = 0.0002$, $S(c_3) = 0.0049, S$ $(c_4) = 0.0007$	$\daleth_1 > \daleth_3 > \daleth_4 > \daleth_2$
	If $\mathfrak{z}(r) = -\log\left((2-r)/r\right)$ then ATSFEWA turns into TSFEWA	$S(c_1) = 0.0030$, $S(c_2) = 0.00000014$, $S(c_3) = 0.0002313$, $S(c_4) = 0.0036$	$\daleth_4 > \daleth_1 > \daleth_3 > \daleth_2$
Mahnaz et al. (2022)	If $\mathfrak{z}(r) = -\log\left(\frac{(\theta-1)}{(\theta^r-1)}\right)$ then ATSFFWA turns into TSFFWA	$S(c_1) = 0.06588$, $S(c_2) = 0.00027$, $S(c_3) = 0.005542$, $S(c_4) = 0.000801$	$\daleth_1 > \daleth_3 > \daleth_4 > \daleth_2$
	If $\mathfrak{z}(r) = -\log\left(\frac{(\theta-1)}{(\theta^r-1)}\right)$ then ATSFFWG turns into TSFFWG	$S(c_1) = 0.0033$, $S(c_2) = 0.000001$, $S(c_3) = 0.00023$, $S(c_4) = 0.0037$	$\daleth_4 > \daleth_1 > \daleth_3 > \daleth_2$
Ullah et al. (2020)	TSFHWA	$S(c_1) = 0.1847$, $S(c_2) = -0.0012$, $S(c_3) = -0.0022$, $S(c_4) = -0.0041$	$\daleth_1 > \daleth_2 > \daleth_3 > \daleth_4$
	TSFHWG	$S(c_1) = 0.01230$, $S(c_2) = -0.0085$, $S(c_3) = 0.0003$, $S(c_4) = -0.0010$	$\daleth_1 > \daleth_3 > \daleth_4 > \daleth_2$
Mahmood et al. (2021)	DTSFPWA	$S(c_1) = -0.1600, S(c_2) = -0.5318$, $S(c_3) = -0.2066, S(c_4) = 0.7198$	$\daleth_1 > \daleth_3 > \daleth_2 > \daleth_4$
	DTSFPWG	$S(c_1) = -0.1801$, $S(c_2) = -0.6715$, $S(c_3) = -0.2212, S$ $(c_4) = 0.7804$	$\daleth_1 > \daleth_3 > \daleth_2 > \daleth_4$
Jiang et al. (2018);	IFS WA IFS WG	Not applicable	Unable to specify
Wei and Lu (2018)	PyFS WA PyFS WA	Not applicable	Unable to specify
Liu and Wang (2018)	q-ROFWA q-ROFWG	Not applicable	Unable to specify

A small briefing of aggregated findings of proposed AOs with other existing AOs in Table 5. is represented gematrically in Figure 2. For more clarity, the outcomes of Table 5 are discussed in Figure 2. In Table 5. we deeply analyze our aggregated findings with Munir et al. (2020), Ullah et al. (2020)**,** Ullah et al. (2020a) and Mahmood et al. (2021a) by applying the AOs described in these references in our proposed example. Since the proposed work of Munir et al. (2020), Ullah et al. (2020)**,** Ullah et al. (2020a) and Mahmood et al. (2021), depends upon the Algebraic sum and product operational laws while our proposed AOs depend upon the changeability of generator. Due to this fact, we believe that our proposed AOs are more reliable than other present AOs.

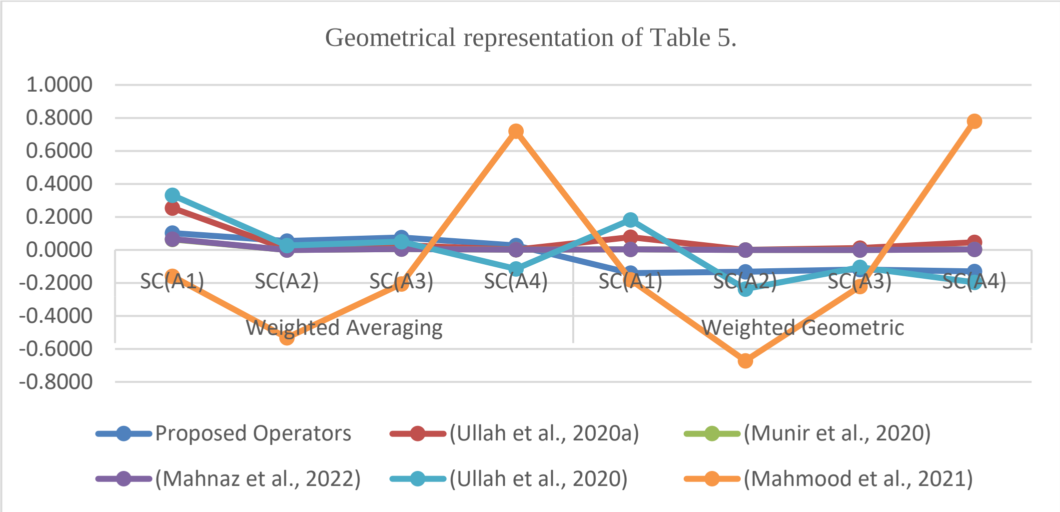

Figure 2. The above graph represents the geometrical view of the comparative analysis, whereas the lines in the graph depicted the score value of the AOs. The aggregation findings given in row 1 are from Mahmood et al. (2021), the aggregation findings given in row 2 are from Ullah et al. (2020a), the aggregation findings given in row 3 are from Ullah et al. (2020), while the aggregation findings are given in row 4 are from Munir et al. (2020).

9. Advantages

ATN and ATCN are very valuable and feasible to evaluate the family of information into a singleton set, because it is the general form of all AOs such as averaging/geometric, Einstein, Hamacher, and frank AOs to compute with the help of algebraic, Einstein, Frank and Hamacher TN and TCN. With the help of different values of function TN and TCN, we can easily obtain these all operators from our one proposed operator, called Archimedean AOs. To enhance the quality and worth of the proposed idea, we describe some special cases of the proposed operators by putting some different values of a function ATN and ATCN, such as:

- **(Hamacher)** If $\mathfrak{z}(r) = -\log\big((\theta + (1-\theta))/r\big), \theta > 0$, then the ATSFWA and ATSFWG operators have reduced to the TSFHWA and TSFHWG operators as defined by Ullah et al (Ullah, Mahmood, and Garg 2020).
- **(Algebraic)** If $\mathfrak{z}(r) = -\log r$, the ATSFWA and ATSFWG operator can be reduced to the TSFWA, and the TSFWG operator is defined by Ullah et al. (Ullah et al. 2020).
- **(Einstein)** If $\mathfrak{z}(r) = -\log\big((2-r)/r\big)$, then ATSFEWA and ATSFEWG operator reduces in the TSFEWA and TSFEWG operator is defined by Munir et al. (Munir et al. 2020).
- **(Frank)** If $\mathfrak{z}(r) = -\log\left(\frac{(\theta-1)}{(\theta^r-1)}\right), \theta > 1$, then the ATSFFWA and ATSFFWG operator turn into the TSFFWA, and the TSFFWG operator is defined by Mahnaz et al. (Mahnaz et al. 2022).

10. Conclusion

For selecting the best preferences, decision-making is valuable and critical technique. The highly notable key points of the analysis are discussed below:

Firstly, discussed the Archimedean operational laws for TSFS and justify them with the help of a numerical example. Diagnosed the theory of ATSFWA, ATSFWG, ATSFOWA, ATSFOWG, ATSFHWA, and ATSFHWG. Some axioms ("Boundedness, Monotonicity, and Idempotency") and the findings of the

aggregated approaches were discussed. To demonstrate the MADM approach on the bases of given TSF information and also discussed the comparative study with other existing prevailing AOs. In this article, geometrical despeciation of proposed information has also discussed the purpose of better understanding.
The following are upcoming aspects of the work:

We aim to apply the proposed technique in complex TSFs power AOs (Khan et al., 2022a), and interval-valued TSF frank AOs (Hussain et al., 2022).

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