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# The simplest amplitude-period formula for non-conservative oscillators

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**ABSTRACT**

The simplest frequency formulation for conservative oscillators was proposed in 2019 (Results Phys 2019;15:102546). However, it becomes invalid for non-conservative oscillators. This work suggests the simplest amplitude-period formulation for non-conservative oscillators. The existence of a periodic solution of a second-order ordinary differential equation is given, and the periodic orbits are easily obtained. To the best of the authors' knowledge, such a powerful result is not available in the literature, providing a tool to determining periodic orbits/limit cycles in the most general scenario.



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## 1. Introduction

Oscillations are an important aspect of dynamic behavior encountered in various fields. Every dynamic system exhibits oscillations of some kind. The most intuitive and obvious oscillations are in the field of mechanics, but they are also regularly encountered in other fields, such as electromagnetism, logistics (stock), economy (business cycles), biology (population cycles) – to name but a few of them. Proper mathematical description and characterization of oscillations are therefore of utmost importance.

This work is focused on mechanical oscillations. Oscillation in mechanics implies a repetitive motion, usually about some central position, which is denoted as the point of equilibrium. Consideration of mechanical oscillations ranges from rather simple single mass undamped free oscillations (Xiao et al., 2011) up to the damped and forced oscillations of complex mechanical structures with an enormously large number of degrees of freedom (Ma et al., 2010; Devillanova et al., 2016; Ghorbaniparvar et al., 2017). Modal analysis (Aoyama et al., 2019; Favarelliet al., 2021) is one of the most important engineering analyses. As a result, it delivers eigenfrequencies and eigenmodes of complex structures, which are needed to characterize the structural dynamics. Comparison between experimental and numerical modal analysis based on the Finite Element Method (FEM) is used on regular basis to check the quality of FE models and perform FE model updating (Tigh Kuchak et al., 2019; Kuchak et al., 2021). Furthermore, all real structures are non-conservative systems as damping is an inevitable part of their dynamic behavior that gives rise to energy dissipation. But in most cases, modal analysis of engineering structures is performed considering undamped systems. By neglecting damping, the accuracy of the obtained results is jeopardized. Despite of that fact, quite often engineers accept such a result in order to have a highly efficient simulation. The reason is in the fact that inclusion of damping in such an analysis would produce results in form of complex numbers, which is not only computationally more tedious, but also more difficult to interpret.

On the other hand, as already emphasized, in reality it is dealt with non-conservative systems and, therefore, it would be crucial to have a mathematical apparatus that can be used to characterize the non-conservative systems in an efficient manner. In 2019, Ji-Huan He (2019) suggested the simplest amplitude-frequency formula for conservative oscillators. The main purpose of this communication is to give a new amplitude-frequency's formulas for non-conservative oscillators (Gelfand & Fomin, 1963; De Brujin, 2010). The existence of the first computable integrals for periodic orbits was suggested (García, 2019), and the application to the case reported by Mickens (2006) is the initial key to develop a general amplitude-frequency formula for non-conservative oscillators. To the best of the authors' knowledge, no other available formula can be found in literature.

## 2. Amplitude-period formula

The main purpose of this communication is the improvement of the main theorem reported in García (2019), removing a condition along with the proof's simplification and presenting a new amplitude-frequency's formula for non-conservative oscillators.

**Theorem 1.** A second order ODE: $\ddot{x}(t)=f(x(t),\dot{x}(t))$, $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$, possesses a periodic orbit:$\{x(0)=A\in\mathbb{R}^{+},x(T)=A,\dot{x}(0)=0\}$, if and only if there exists a function $\dot{x}(t)=\phi(x)\in\mathbb{C}^{1}(\mathbb{R})$, such that: $\frac{d\phi(x)}{dx}=\frac{f(x,\phi(x))}{\phi(x)},\phi(A)=0$.

**Proof .** We first prove its necessity.

If there exists $\dot{x}(t)=\phi(x)\in\mathbb{C}^{1}(\mathbb{R})$, such that: $\frac{d\phi(x)}{dx}=\frac{f(x,\phi(x))}{\phi(x)},\phi(A)=0$, then: $\ddot{x}(t)=f(x(t),\phi(x))$. This new ODE is in fact a conservative oscillator (see for instance (Mickens, 2010)).

Now we prove its sufficiency. Performing an asymptotic expansion for $\ddot{x}(t)=f(x(t),\dot{x}(t))$, using an arbitrary bounded function $\zeta\in\mathbb{C}(\mathbb{R}^{1})$:

$$f(x,\dot{x})\sim f(x+\eta\cdot\zeta,\dot{x}+\eta\cdot\dot{\zeta})+\left.\frac{\partial f(x,\dot{x})}{\partial x}\right|_{\{x+\eta\cdot\zeta,x+\eta\cdot\dot{\zeta}\}}\cdot[x-(x+\eta\cdot\zeta)]+\left.\frac{\partial f(x,\dot{x})}{\partial\dot{x}}\right|_{\{x+\eta\cdot\zeta,x+\eta\cdot\dot{\zeta}\}}\cdot[\dot{x}-(\dot{x}+\eta\cdot\dot{\zeta})],(\eta\rightarrow0)$$

Integrating and taking into account the periodic orbits' hypothesis:

$$\int_0^T f(x,\dot{x})\cdot dt=0\sim\int_0^T f(x+\eta\cdot\zeta,\dot{x}+\eta\cdot\dot{\zeta})\cdot dt+\int_0^T\left.\frac{\partial f(x,\dot{x})}{\partial x}\right|_{\{x+\eta\cdot\zeta,x+\eta\cdot\dot{\zeta}\}}\cdot[x-(x+\eta\cdot\zeta)]\cdot dt+\int_0^T\left.\frac{\partial f(x,\dot{x})}{\partial\dot{x}}\right|_{\{x+\eta\cdot\zeta,x+\eta\cdot\dot{\zeta}\}}\cdot[\dot{x}-(\dot{x}+\eta\cdot\dot{\zeta})]\cdot dt=0,(\eta\rightarrow0)$$

Equivalently:

$$\int_0^T\eta\cdot\zeta(t)\cdot\left[\frac{d}{dt}\left(\frac{\partial f(x,\dot{x})}{\partial\dot{x}}\right)+\frac{\partial f(x,\dot{x})}{\partial x}\right]=0,(\eta\rightarrow0)$$

Applying the fundamental lemma of calculus of variations for a test function $\eta\cdot\zeta(t)$ (see for instance (Landau & Lifshitz, 1982), page 9):

$$\frac{d}{dt}\left(\frac{\partial f(x,\dot{x})}{\partial\dot{x}}\right)+\frac{\partial f(x,\dot{x})}{\partial x}=0$$

This equation is a necessary condition for the first variation of the following functional:

$$\min_{x(t)\in\mathbb{R}}\underbrace{\int_0^t f(\sigma,\dot{\sigma})\cdot d\sigma}_{J}$$

$$\text{such that:}\quad\begin{cases}\ddot{x}=f(x,\dot{x})\\ x(0)=A\end{cases}$$

In summary, the first variation $\delta J = 0$ for every periodic trajectory $\{x = \xi_1(t, A, \dot{x}(0)), \dot{x} = \xi_2(t, A, \dot{x}(0))\}$ of $\ddot{x}(t) = f(x, \dot{x})$, i.e.:

$$\frac{\partial}{\partial \dot{x}(0)} \int_0^T f\big(\xi_1(\sigma, A, \dot{x}(0)), \xi_2(\sigma, A, \dot{x}(0))\big) \cdot d\sigma = 0, \forall t \in [0, T]$$

Then:

$$\frac{\partial}{\partial \dot{x}(0)} f\big(\xi_1(\sigma, A, \dot{x}(0)), \xi_2(\sigma, A, \dot{x}(0))\big) = 0, \forall t \in [0, T]$$

This conclusion shows that actually: $f(x, \dot{x}) = f(x)$, in other words, it is a conservative oscillator with $\phi(x) \in \mathbb{C}^1(\mathbb{R})$, such that: $\dot{x}(t) = \phi(x(t))$. This completes the proof.

Finally, specializing this result to oscillators: $\ddot{x} = f_1(x) \cdot f_2(\dot{x})$, we have the following corollary.

Corollary 1: A second order ODE: $\ddot{x}(t) = f(x(t), \dot{x}(t))$, $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, possesses a periodic orbit: $\{x(0) = A \in \mathbb{R}^+, x(T) = A, \dot{x}(0) = 0\}$, with an amplitude-period formula:

$$\begin{cases} \int_0^{\phi(x)} \frac{\phi}{f_2(\phi)} \cdot d\phi = \int_A^x f_1(x) \cdot dx \\ T = 4 \cdot \int_0^A \frac{1}{\phi(x)} \cdot dx \end{cases}$$

Applying the corollary to Mickens' oscillator: $\ddot{x} = -x \cdot (1 + \dot{x}^2)$, we have

$$\int_0^{\phi(x)} \frac{\phi}{1 + \phi^2} \cdot d\phi = \int_A^x (-x) dx$$

and

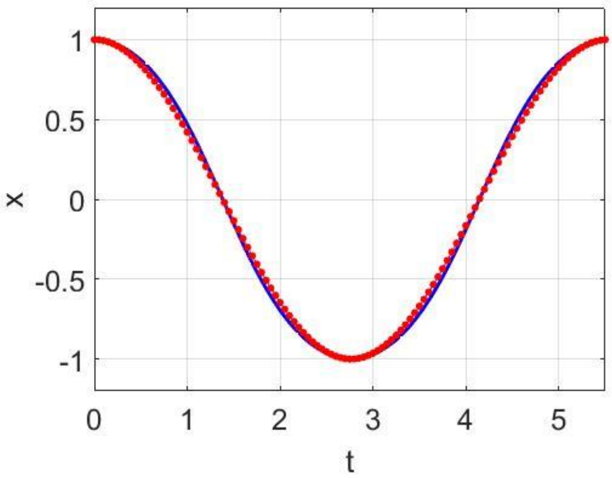
$$\phi = \sqrt{e^{A^2 - x^2} - 1}$$

The amplitude-period relationship is obtained as follows

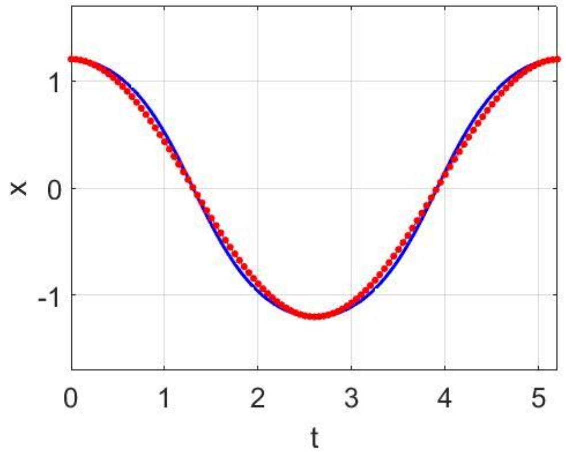
$$T = 4 \cdot \int_0^A \frac{dx}{\sqrt{e^{A^2 - x^2} - 1}}$$

Notice that this formula, is not more than the exact amplitude-frequency formula obtained in (Mickens, 2006).

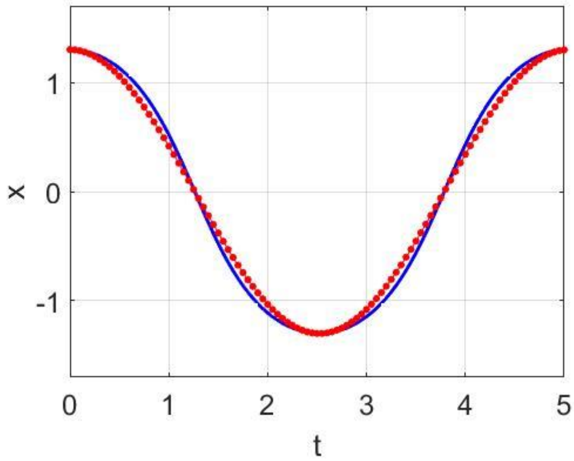
Fig.1 shows the accuracy of the approximate solution.



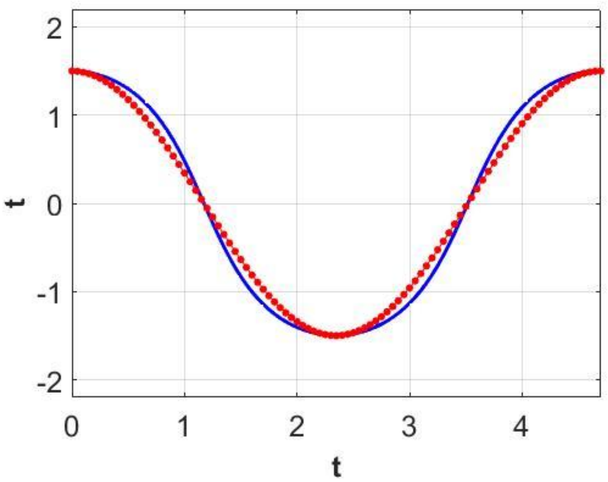

(a)



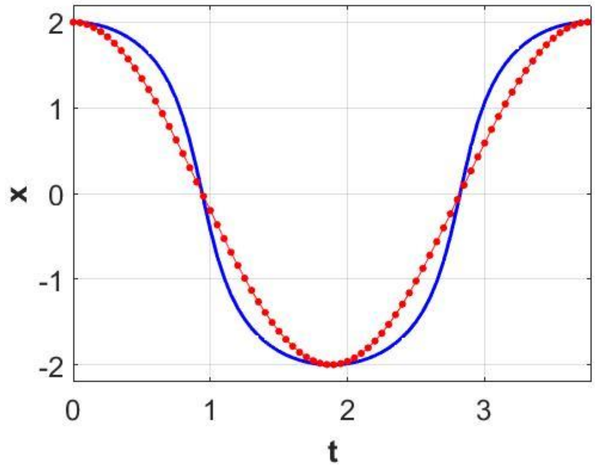

(b)




(c)




(d)




(e)

Figure 1. Comparison of the exact solution (continuous blue line) and the approximate one (red line with dots): (a) A=1; (b) A=1.2; (c) A=1.3; (d) A=1.5; (e) A=2

## 3. Conclusions

Proper mathematical description and characterization of mechanical oscillations are of utmost importance in the field of mechanical engineering. Most of the developed and well-established mathematical formulas pertain to conservative systems, while the real systems are actually non-conservative. For this reason, this short communication sets an important objective of providing a few formulas for non-conservative systems.

A novel first computable integral to reduce a second order non-conservative oscillator: $\ddot{x}(t) = f(x(t), \dot{x}(t))$, $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to a conservative one: $\ddot{x}(t) = f(x(t), \phi(x(t)))$ is proved.

To the best of the author's knowledge, this result, which includes the possibility to account for periodic orbits computing a reduced first order ODE: $\dot{x} = \phi(x)$, has not been published yet.

The specialization of the results in this paper to ODE's: $\ddot{x} = f_1(x) \cdot f_2(\dot{x})$ provides an amplitude-period formula for non-conservative oscillators.

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