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Suitability for coding of the Colebrook's flow friction relation expressed by symbolic regression approximations of the Wright-ω function

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ABSTRACT

This article analyses a form of the empirical Colebrook's pipe flow friction equation given originally by the Lambert W-function and recently also by the Wright ω-function. These special functions are used to explicitly express the unknown flow friction factor of the Colebrook equation, which is in its classical formulation given implicitly. Explicit approximations of the Colebrook equation based on approximations of the Wright ω-function given by an asymptotic expansion and symbolic regression were analyzed in respect of speed and accuracy. Numerical experiments on 8 million Sobol's quasi-Monte points clearly show that also both approaches lead to approximately the same complexity in terms of speed of execution in computers. However, the relative error of the developed symbolic regression-based approximations is reduced significantly, in comparison with the classical basic asymptotic expansion. These numerical results indicate promising results of artificial intelligence (symbolic regression) for developing fast and accurate explicit approximations.

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1. Introduction

Relation developed by Colebrook (1939) is widely used for calculation of friction factor in pipes, Eq. (1):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot log_{10}\left(\frac{2.51}{Re} \cdot \frac{1}{\sqrt{\lambda}} + \frac{\varepsilon}{3.71}\right) \quad (1)$$

In the Colebrook equation λ is the unknown Darcy's flow friction factor, while Re is the Reynolds number and ε is the relative roughness of inner pipe surface (all three quantities are dimensionless). The flow friction factor λ is locked in an implicit form through a logarithmic expression. Domains of the input parameters used in engineering practice are $4000 < Re < 10^8$ and $0 < \varepsilon < 0.05$, while for the output parameter $0 < \lambda < 0.088$. The Reynolds number is an important dimensionless quantity in fluid mechanics which is used to predict transition from laminar, sheet-like flow in parallel layers to turbulent flow which is characterized with unsteady vortices, while the relative roughness ε is calculated as the absolute roughness of an inner pipe surface divided by the inside diameter of a pipe, where the absolute roughness represents the average the height across the microscopic peaks and valleys above the laminar sub-layer of fluid (see Fig. 1). It is not easy to estimate actual value of absolute roughness of a pipe material (Guo et al. 2020), but it is from around 0.0025mm for glass and plastic which are practically smooth, 0.025mm new smooth concrete, around 0.15mm for casted iron, 0.25mm for coarse concrete, around 0.5mm for rusted steel, etc. (Moody 1944).

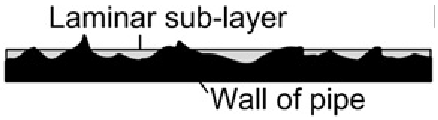

Figure 1. Absolute roughness of inner pipe surface: average height across the microscopic peaks and val-leys above the laminar sub-layer of fluid.

Depending on thickness of the laminar sub-layer which can cover partially or entirely the protrusions of pipe material, value of hydraulic roughness can varies also depending on the type of fluid but also on the Reynolds number where for the laminar flow (laminar from occurs for $Re < 4000$, but can be also for the higher values of Re; on the other hand, turbulent flow cannot exist for $Re < 4000$) all materials are practically smooth, where for the fully developed turbulent flow (high values of Re), laminar sub-layer does not exist and physical roughness of the surface is equal to the hydraulic roughness (Brkić 2012a).

The Colebrook equation is of empirical nature (Colebrook and White 1937). Its graphical interpretation is given by Moody diagram (Moody 1944). Number of explicit approximations of the Colebrook equation exists (Assuncao et al. 2020, Brkić 2011ab, Brkić and Ćojbašić 2017), and here we will offer few very accurate and computationally efficient which are based on the Wright ω-function, a cognate of the Lambert W-function (Corless et al. 1996). Thanks to these special functions, it is capable to transform expressions from their implicit in an explicit form, which is suitable for further processing (Brkić 2011bc, Brkić 2012c, Brkić and Praks 2019, Praks and Brkić 2020).

2. Mathematics behind the proposed solution

2.1. Lambert W-function and Wright ω-function

The Lambert W-function (Corless et al. 1996, Barry et al. 2000, Fukushima 2020a) is defined as the inverse function of $f(W) = W \cdot e^W$, whereas the Wright ω-function with x as the argument solves the equation $y + ln(y) = x$. Thus, the Wright ω-function (Corless and Jeffrey 2002) is a cognate of the Lambert W-function, and here it is used to transform the Colebrook equation from the implicitly given form in respect of the unknown variable to the explicit form (Brkić and Praks 2019, Praks and Brkić 2020). Additionally, about application of the Lambert W-function to the Colebrook equation can be seen in Alfaro-Guerra et al. 2020, Vatankhah 2018, Biberg 2017, Mikata and Walczak 2016, Viccione and Tibullo 2012, etc.

However, as the further evaluation of the Wright ω-function can be only approximate, approximations of the Wright ω-function are developed for the various purposes and hence they are very well tested (Corless and Jeffrey 2002, Fukushima 2020b).

The Wright ω-function is the Lambert W-function with a shifted argument, and here is used for the fact that such shifted argument is not fast-growing as in the case of the Lambert W-function (Brkić 2012d). If the Lambert W-function is used for solving the Colebrook equation, an overflow error will occur in about half cases used in engineering practice (Sonnad and Goudar 2002), as the required term $W(e^x)$ cannot be accurately stored in registers. However, some software packages (e.g. Matlab) in such cases use a numerically stable the Wright ω-function instead of the Lambert W-function with the purpose to avoid the overflow error by the equivalence $W(e^x) = \omega(x)$, where e.g. for $x = 500$, $y = W(e^{500}) = \omega(500) = 493.7978$.

2.2. Asymptotic expansion

We will test approximations of $y = \omega(x) - x$ given by the asymptotic expansion provided by Corless and Jeffrey (2002).

The used asymptotic expansion series are given in Table 1 denoted by the symbol AE and summarized in the first four lines.

2.3. Symbolic regression

Symbolic regression is based on artificial intelligence and it is capable to find approximations for the certain function given by data sets. The main advantage of such approach is that the symbolic expression is found by an artificial intelligence tool and it is not prescribed in advance. The user only specifies building blocks of the symbolic formula (for example, arithmetic operations, logarithms, etc.). Here is used software tool Eureqa (Dubčáková 2011).

The developed symbolic regression formulas are in Table 1 denoted by the symbol SR (additional formula are in Conclusions).

2.4. Matlab built-in implementation and WrightOmegaq library

In Matlab, the built-in Wright ω-function is denoted by the command wrightOmega. In Table 1, this Matlab built-in function is represented by the symbol “Matlab built-in”. However, Table 1 also shows the results of an alternative open-source implementation of the Wright ω-function, which is represented by the symbol “wrightOmegaq”. The “wrightOmegaq” library works well also in GNU Octave. Table 1 presents results of two variants of the formula: exact constants and approximated constants. The “exact constants” is the exact solution given in Eq. (2), whereas the “approximated constants” is the proposed approximation suitable for engineering practice given in Eq. (3).

The aim of this comparison is to demonstrate how reducing the number of digits used in Eq. (3) will reduce the accuracy of results.

3. Exact explicit solution of the Colebrook equation

The Colebrook equation in term of the Wright ω-function can be given as Eq. (2):

$$\begin{cases} \frac{1}{\sqrt{\lambda}} = \frac{C}{2.51} \cdot (B + y) \\ C = \frac{2 \cdot 2.51}{ln(10)} \\ A = \frac{Re}{C} \cdot \frac{\varepsilon}{3.71} \\ B = ln(Re) - ln(C) \\ x = A + B \\ y = \omega(x) - x \end{cases} \tag{2}$$

Series about infinity of $\omega(x) - x = -ln(x) + \sum_{l\geq 0} \sum_{m\geq 0} c_{lm} \frac{ln^m(x)}{x^{l+m}}$ is defined in terms of Striling cycle numbers, where $c_{lm} = (-1)^l \begin{bmatrix} l+m \\ l+1 \end{bmatrix} / m!$, where m and l are positive integer numbers (Rollmann and Spindler 2015). For the Colebrook equation, $\omega(x) - x$ is strictly monotonic decreasing as can be seen from Fig. 2, where x for the practical engineering interest goes for the Colebrook’s equation from 7 to 619, while the diagram is plotted in Matlab as:

```
syms x
assume(x>=7 & x<=619)
f=wrightOmega(x)-x
fplot(f, [7,619])
```

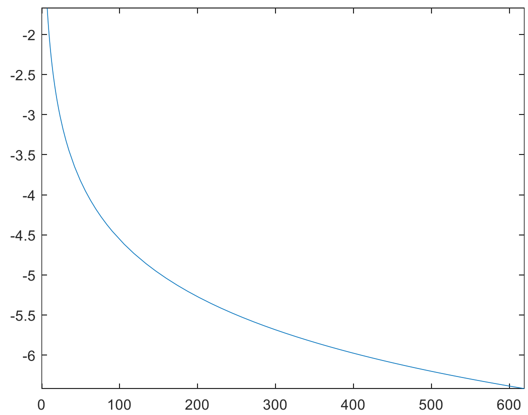

Figure 2. Strictly monotonic decreasing of $y = \omega(x) - x$ for the Colebrook equation; the horizontal axis represents x and the vertical y – this figure is adapted from Praks and Brkić (2020).

One possible simplification based in the form of approximation suitable for engineering use can be, Eq. (3):

$$\begin{cases} \frac{1}{\sqrt{\lambda}} \approx 0.868589 \cdot (B + y) \\ A \approx \frac{Re \cdot \varepsilon}{8.0878} \\ B \approx ln(Re) - 0.7794 \\ x \approx A + B \end{cases} \quad (3)$$

Where $y = \omega(x) - x$ is approximated in Table 1 in Matlab notation. The column 'time in sec' is given for execution of the Colebrook equation for 8 million samples generated using the Sobol's Quasi Monte Carlo algorithm, which covers the entire region of interest very efficiently (Sobol et al. 1992).

Table 1. Comparison of approximations of $y = \omega(x) - x$ with related error and speed of execution; Asymptotic expansion (AE), Symbolic regression (SR).

Type of $\omega(x)$ approximation	Approximation of y	Relative error in %	Ratio	Time in sec
AE	y=lnx./x-lnx	1.52E-01	1	0.7
AE	y=lnx./x-lnx+0.000818	1.36E-01	1.1	0.7
AE	y=lnx./x-lnx+0.5.*lnx.*(lnx-2)./x.^2	1.18E-01	1.3	0.9
AE	y=lnx./x-lnx+0.5.*lnx.*(lnx-2)./x.^2-0.002	9.61E-02	1.6	0.9
SR	y=(1.038*lnx)./(x + 0.332) - lnx	5.22E-02	2.9	0.7
SR	y=(1.0119*lnx)./x - lnx + (lnx - 2.3849)./x.^2	8.45E-03	18.0	0.8
wrightOmegaq	y=wrightOmegaq(x)-x (approximated constants)	2.49E-03	61	3.8
wrightOmegaq	y=wrightOmegaq(x)-x (exact constants)	0		2.7
Matlab built-in	y=wrightOmega(x)-x	0		5158

Results from Table 1 demonstrate that the speed of both used approximations (the asymptotic expansion based vs. symbolic regression based) is approximately the same, as the computation time varies from 0.7 sec to 0.9 sec. In order to quantify the improvement of the approximations from Table 1, we computed the relative precision ratio between the basic approximation y=lnx./x-lnx and the given approximations. For this reason, the basic approximation y=lnx./x-lnx has the improvement equal to one, where the first symbolic regression approximation from Table 1 has improvement 2.9, while the second symbolic regression approximation has improvement 18, as the relative precision is reduced from 1.52E-01% to 8.45E-03%. Moreover, we can see that the symbolic regression approximations have approximately the same speed as the original asymptotic regression approximation y=lnx./x-lnx (0.7sec and 0.8 sec vs. 0.7 sec).

Numerical experiments on 8 million Sobol's quasi-Monte points clearly show that although both approaches lead to the approximately same complexity (the computer speed is similar), the relative error of the novel symbolic regression approximation is reduced 18 times, if we compare it with the original asymptotic regression approxima-tion y=lnx./x-lnx. We can also clearly observe from the Table 1, that the Wright ω-function built-in in Matlab is extremely slow, in comparison with the open-source library wrightOmegaq: The Wright ω-function of Matlab requires 5158 seconds for the benchmark of Table 1, whereas the wrightOmegaq requires less than 4 seconds. The corresponding speed-up ratio is 5158/4~1289.

We can also see from the Table 1 that it make sense to create fast approximations of the wrightOmegaq library: The exact solution given by the wrightOmegaq library requires 2.7-3.8 seconds, whereas all the here mentioned fast approximations always require less than 1 second.

4. Conclusions

Numerical experiments on 8 million Sobol's quasi-Monte points clearly show that both approaches, asymptotic expansion and symbolic regression, to make approximate relations for the Colebrook equation for flow friction transformed through the Wright ω-function, lead to the approximately same complexity of the obtained approximations (computer speed is also similar). However, the relative error of the two developed symbolic regression approximations is reduced by factor 3 and 18, respectively, in comparison with the classical asymptotic expansion. These numerical results indicate promising results of artificial intelligence (symbolic regression) for the area of fast and accurate explicit approximations. On the other hand, although accurate, build-in Matlab function for the Wright ω-function is extremely slow. Numerical experiments in Table 1 also show that the alternative open-source implementation given by the library "wrightOmegaq" is much faster than the Matlab build-in implementation (Horchler 2017).

As reported in Praks and Brkić (2020), as extension to the results from Table 1, approximation y=lnx./(x – 0.5564*lnx + 1.207). - lnx, is to date the most accurate.

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