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Stress tensor in the linear viscoelastic incompressible half-space beneath axisymmetric bodies in normal contact

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ABSTRACT

We present a simple method to determine the stress state in a linear viscoelastic half-space in frictionless unilateral normal contact with an axisymmetric rigid indenter for arbitrary loading histories. The procedure consists in the superposition of elastic solutions for flat punch or parabolic contact along the line load within the framework of the method of dimensionality reduction. We provide simple expressions for all stress components and the hydrostatic pressure gradient, which is of particular interest in the biomechanical field. The method brings advantages especially in its numerical application, since no convolution in time over the entire contact history is necessary.

1 Introduction

Lee (1955) firstly proposed a method to derive the viscoelastic solution for the frictionless normal contact problem from the equivalent elastic problem by convolution with the time dependent relaxation function, i.e. a multiplication in the Laplace-space – with the restriction that the contact area monotonically increases with time. This idea, known as the "elastic-viscoelastic correspondence principle", was later generalized by Radok (1957), before Lee and Radok (1960) derived the solution for the contact of a parabolic indenter and a viscoelastic half-space. The restriction of the monotonically increasing contact area was partially released by Hunter (1960) and Graham (1965) to the case where it has a single maximum and no minimum; the solutions for arbitrary loading histories were ultimately found by Graham (1967) and Ting (1966, 1968).

Easy and fast solving of axisymmetric contact problems between viscoelastic bodies is also possible using the method of dimensionality reduction (MDR), developed by Popov and Heß (2014) and originally introduced for elastic media. The MDR consists in converting an originally three-dimensional contact problem to a one-dimensional one with a Winkler foundation, which, however, has the same contact properties. Thus, the complexity of the original contact problem is reduced. While the macroscopic parameters (i.e. contact force, indentation depth and contact radius) are identical in both spaces, a set of transformation rules links load distributions, displacements and the indenter profiles between spaces. It has quite recently been shown that the full stress state beneath the contact surface in axisymmetric elastic normal contact problems can be determined by superposition of flat punch (Forsbach 2020) or parabolic (Willert 2021a) solutions according to the transformed shape of the indenter in the 1D-space within the framework of the MDR.

When it comes to viscoelastic media the MDR was introduced based on the Lee and Radok elastic-viscoelastic correspondence principle (Popov and Heß 2014; Popov et al. 2018, 2019) – inheriting of its restriction, i.e. the monotonically increasing contact radius. However Argatov and Popov (2016) extended the MDR to the case where the contact radius has a single maximum while at the same time solving the rebound indentation problem, an important application of this solution to material testing. The big advantage of the viscoelastic

MDR, especially when it comes to its numerical application, lies in the avoidance of the convolution from Lee and Radok's solution, since it is sufficient to follow the displacement of rheological elements, forming the Winkler foundation and representing a certain material behavior, to yield the solution of the contact problem – until now the solution of pressure and displacements in the contact plane. In the following we will present a method on the basis of the MDR to determine the full stress state beneath the surface for arbitrary contact histories, without loosing the advantage of avoiding the convolution. As the procedure is applicable with any linear operation on the stress components, we will also provide compact expressions for the hydrostatic pressure and its gradient. The latter can be of interest for biomechanical applications, e.g. for the mechanical modelling of biological joints and their regeneration, where the hydrostatic pressure gradient in the viscoelastic articular cartilage in contact plays an important role (Leroy 2023; Popov et al. 2022).

After a brief overview of the viscoelastic MDR, we derive in the subsequent section two equations determining the stress state in the linear viscoelastic incompressible half-space beneath axisymmetric bodies in normal contact by superposition of elastic flat-plunch or parabolic contact solutions. As an illustrative example, we determine the hydrostatic pressure gradient in a viscoelastic half-space impacted by a rigid sphere before a short conclusion closes the manuscript.

2 Method of dimensionality reduction for viscoelastic contacts

Let us first consider a rigid axisysmmetric frictionless indenter with a shape defined by the function $f(r)$ (with $f(0) = 0$ and $f'(r) > 0$ for $r > 0$, the prime denoting a derivative in space) pressed into a linear elastic half-space (with elastic modulus E_0 and Poisson's ratio ν) in the normal direction. The origin of the cylindrical coordinates system (r, z) is located in the first point of contact, the z-axis pointing into the half-space. Within the framework of the MDR this three-dimensional problem can be transformed into a one-dimensional contact problem of a profile $g(x)$ being pressed into a Winkler foundation with a stiffness per unit length of $E_0^* = E_0/(1-\nu^2)$. The relationship between g and f ist given by an Abel-like integral:

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r. \tag{1}$$

This immediately gives the relationship between the contact radius a and the indentation depth d, i.e. $g(a) = d$. In analogy to the pressure distribution in 3D, in the 1D-space one can define a line load

$$q_\mathrm{e}(x) = E_0^*(g(a) - g(x))H(a - x), \tag{2}$$

– where $H(a - x)$ is the Heaviside step function – from which the three-dimensional pressure distribution can be deduced (Popov et al. 2019):

$$p_\mathrm{e}(r) = -\frac{1}{\pi} \int_r^\infty \frac{q'_\mathrm{e}(x)}{\sqrt{x^2 - r^2}} \mathrm{d}x. \tag{3}$$

Note that in the numerical application of the MDR, transformations (1) and (3) can be performed very efficiently using FFT (Willert 2021b).

As shown by Forsbach (2020) not only the pressure distribution in the surface but also the full stress state inside the elastic half-space can be obtained for an arbitrary axisymmetric indenter based on the profile g by superposition of flat punch solutions:

$$\sigma_{ij}^{\mathrm{AS,e}}(r, z, a) = \int_0^a \frac{\sigma_{ij}^{\mathrm{FP}}(r, z, \tilde{a})}{d^{\mathrm{FP}}} g'(\tilde{a}) \mathrm{d}\tilde{a}, \tag{4}$$

where $\sigma_{ij}^{\mathrm{FP}}(r, z, \tilde{a})$ is the stress field generated by the indentation of the elastic half-space by a rigid cylindrical flat punch with radius $\tilde{a}$ and with indentation depth d^{FP}, recently given in compact form by Willert (2023).

The MDR is also applicable to linear viscoelastic media. Let us from now on consider an incompressible, linear viscoelastic half-space with relaxation modulus $E(t) = E_0\psi(t)$, where $\psi(t)$ is the normalized relaxation function. If the radius of contact posesses a single maximum (and no minimum), Argatov and Popov (2016) showed that the line load $q(x, t)$ is given by the following integral in time over the elastic solution, which becomes a function of time, since $a = a(t)$:

$$q(x, t) = \int_0^{t_1(t)} \psi(t - \tau) \frac{\partial q_\mathrm{e}(x, \tau)}{\partial \tau} \mathrm{d}\tau, \tag{5}$$

where $t_1(t)$ depends on the phase of the indentation. In the phase of monotonically increasing contact radius, i.e. before reaching the maximum contact radius at time $t_{\max}$, we have

$$t_1(t) = t \text{ for } t \leq t_{\max}, \tag{6}$$

which lets equation (5) take the form according to the elastic-viscoelastic correspondance principle. In the following phase of monotonically decreasing contact radius, $t_1(t)$ is the solution of the equation

$$a(t) = a(t_1(t)), \;\; t_1(t) < t_{\max} \text{ for } t > t_{\max}. \tag{7}$$

Alternatively to evaluating the integral in equation (5), which generally requires the knowledge of the entire loading history, one can also solve equilibrium conditions for rheological elements representing a certain time-dependent material behavior and forming the Winkler foundation to determine the line load q; for details on this procedure see Popov et al. (2018, 2019). This is especially advantageous when solving problems numerically, since not the entire loading history has to be stored, but, for instance using an Euler method, only the last time step. Note that when determining the line load in this way, adhesion can also be easily taken into account using a simple detachment criterion (Popov 2021).

3 Superposition rule

In analogy to the superposition method for the elastic problem we will show that to determine the full stress state beneath the surface of an linear viscoelastic incompressible half-space the same principle applies, but integrating along the line load $q(x,t)$ and not along the profile g:

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = -\int_0^{a(t)} \frac{\sigma_{ij}^{\mathrm{FP}}(r,z,\tilde{a})}{d^{\mathrm{FP}}} \frac{q'(\tilde{a},t)}{E_0^*} \mathrm{d}\tilde{a}. \tag{8}$$

For this let us first insert equation (5) for $q'(\tilde{a},t)$:

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = -\int_0^{a(t)} \frac{\sigma_{ij}^{\mathrm{FP}}(r,z,\tilde{a})}{d^{\mathrm{FP}}} \frac{1}{E_0^*} \int_0^{t_1(t)} \psi(t-\tau) \frac{\partial^2 q_{\mathrm{e}}(\tilde{a},\tau)}{\partial x \partial \tau} \mathrm{d}\tau \mathrm{d}\tilde{a}. \tag{9}$$

The occurring derivative of q_{e} reads:

$$\frac{\partial^2 q_{\mathrm{e}}(x,t)}{\partial x \partial t} = E_0^* \dot{a}(t) \left[\delta(a(t)-x)\left(-g'(a(t)) - g'(x)\right) + \delta'(a(t)-x)\left(g(x) - g(a(t))\right)\right], \tag{10}$$

where $\delta(a(t)-x)$ is the Dirac delta function and the dot is denoting a derivative in time. After inserting it in (9), some shuffling in the equation and evaluating the integral in space, we get:

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = \int_0^{t_1(t)} \psi(t-\tau)\dot{a}(\tau) \frac{\sigma_{ij}^{\mathrm{FP}}(r,z,a(\tau))}{d^{\mathrm{FP}}} g'(a(\tau)) \mathrm{d}\tau. \tag{11}$$

We now compare this result to the convolution-like equation over the elastic solution as a whole to calculate the viscoelastic solution for an incompressible (and only for this case) medium, found at Graham (1965):

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = \int_0^{t_1(t)} \psi(t-\tau) \frac{\partial \sigma_{ij}^{\mathrm{AS,e}}(r,z,a(\tau))}{\partial \tau} \mathrm{d}\tau \tag{12}$$

Inserting the elastic solution (4) and applying the Leibniz integral rule, we get:

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = \int_0^{t_1(t)} \psi(t-\tau) \frac{\partial \left[\int_0^{a(\tau)} \frac{\sigma_{ij}^{\mathrm{FP}}(r,z,\tilde{a})}{d^{\mathrm{FP}}} g'(\tilde{a}) \mathrm{d}\tilde{a}\right]}{\partial \tau} \mathrm{d}\tau \tag{13}$$

$$= \int_0^{t_1(t)} \psi(t-\tau)\dot{a}(\tau) \frac{\sigma_{ij}^{\mathrm{FP}}(r,z,a(\tau))}{d^{\mathrm{FP}}} g'(a(\tau)) \mathrm{d}\tau, \tag{14}$$

which is the same equation as (11) and gives proof of the validity of the superposition rule in equation (8). On the basis of Ting's (1968) general solution, it can be shown in an analogous way that this superposition rule must also be valid for arbitrary contact histories.

Just as shown by Willert (2021a) for the elastic case, based on the stated superposition rule using flat punch solutions (eq. (8)) a superposition procedure using Hertzian solutions for parabolic contact can be derived. We will skip the derivation, which is exactly the same as in the aforementioned paper. The result is:

$$\sigma_{ij}^{\mathrm{AS}}(r,z,a(t)) = \frac{R}{E_0^*}\left[\int_0^{a(t)} \sigma_{ij}^{\mathrm{H}}(r,z,\tilde{a})\left(\frac{q''(\tilde{a},t)}{2\tilde{a}} - \frac{q'(\tilde{a},t)}{2\tilde{a}^2}\right)\mathrm{d}\tilde{a} - \sigma_{ij}^{\mathrm{H}}(r,z,a(t))\frac{q'(a(t),t)}{2a}\right], \tag{15}$$

where $\sigma_{ij}^{\mathrm{H}}(r,z,\tilde{a})$ is the stress field generated by the indentation of the elastic half-space by a rigid parabolic indenter with a shape defined by $f(r) = r^2/(2R)$, with a radius of contact $\tilde{a}$; it is given in explicit form by Huber (1904) or Hamilton (1983).

It is important to note that the described superposition principle is applicable to any linearly operated stress components, e.g. the gradient of hydrostatic pressure $p_{\mathrm{hs}} = -(\sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{zz})/3$ or its gradient. Using the stress fields recently derived by Willert (2023) in the half-space beneath a circular flat punch, we can adapt the superposition rule (4) to this specific case and get:

$$p_{\mathrm{hs}}^{\mathrm{AS}}(r,z,a(t)) = -\frac{1}{\pi}\int_0^{a(t)} \frac{1}{\sqrt{S}}\sin\left(\frac{\pi}{4} - \frac{1}{2}\mathrm{atan}\left(\frac{A}{2z\tilde{a}}\right)\right) q'(\tilde{a},t)\mathrm{d}\tilde{a}, \tag{16}$$

$$\frac{\mathrm{d}p_{\mathrm{hs}}^{\mathrm{AS}}}{\mathrm{d}r}(r,z,a(t)) = -\frac{1}{\pi}\int_0^{a(t)} \frac{r}{\sqrt{S^3}}\sin\left(\frac{3\pi}{4} - \frac{3}{2}\mathrm{atan}\left(\frac{A}{2z\tilde{a}}\right)\right) q'(\tilde{a},t)\mathrm{d}\tilde{a}, \tag{17}$$

$$\frac{\mathrm{d}p_{\mathrm{hs}}^{\mathrm{AS}}}{\mathrm{d}z}(r,z,a(t)) = -\frac{1}{\pi}\int_0^{a(t)} \sqrt{\frac{z^2+\tilde{a}^2}{S^3}}\sin\left(\frac{\pi}{4} + \mathrm{atan}\left(\frac{z}{\tilde{a}}\right) - \frac{3}{2}\mathrm{atan}\left(\frac{A}{2z\tilde{a}}\right)\right) q'(\tilde{a},t)\mathrm{d}\tilde{a}, \tag{18}$$

where $A = r^2 + z^2 - \tilde{a}^2$ and $S = \sqrt{A^2 + 4\tilde{a}^2 z^2}$.

Equations (8) and (15)-(18) provide explicit solutions for three-dimensional stress components and their derivatives (e.g. pressure gradient) based on the solution $q(x,t)$ obtained in the framework of the MDR.

4 Numerical example

In order to illustrate the described procedure, we consider a rigid sphere of radius R and with mass m impacting a viscoelastic half-space in the normal direction with the velocity v_0, which is supposed to be small compared to the characteristic velocity of wave propagation in the medium, so that we have a quasi-static problem. The absolute value of the hydrostatic pressure gradient in the half-space is to be determined, i.e.

$$\left|\vec{\nabla} p_{\mathrm{hs}}\right| = \sqrt{\left(\frac{\mathrm{d}p_{\mathrm{hs}}^{\mathrm{AS}}}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}p_{\mathrm{hs}}^{\mathrm{AS}}}{\mathrm{d}z}\right)^2}. \tag{19}$$

The viscoelastic half-space shall be a ”standard solid” with the normalized relaxation function

$$\psi(t) = \frac{G_\infty + G_1\exp\left(-\frac{G_1}{\eta}t\right)}{G_\infty + G_1}, \tag{20}$$

where $G_\infty + G_1$ is the instantaneous modulus, G_∞ the static modulus and η the shear viscosity. The corresponding rheological model forms the elements of the Winkler foundation when solving the problem in the context of the MDR (see figure 1). For this, firstly the line load q needs to be determined. This is easily achieved by solving the equilibrium conditions of the rheological elements of the foundation. If we introduce a dicretisation in the space coordinate $x_i = i\Delta x$ (denoted by subscript) and one in time $t^j = j\Delta t$ (denoted by superscript) and apply the explicit Euler method, these equilibrium conditions read:

$$s_i^j = \frac{G_1 u_i^j + \frac{\eta}{\Delta t}s_i^{j-1}}{G_1 + \frac{\eta}{\Delta t}}, \tag{21}$$

$$q_i^j = 4\left((G_1 + G_\infty)u_i^j - G_1 s_i^j\right). \tag{22}$$

The displacement u of the Winkler foundation is determined by the time-dependent indentation depth d and

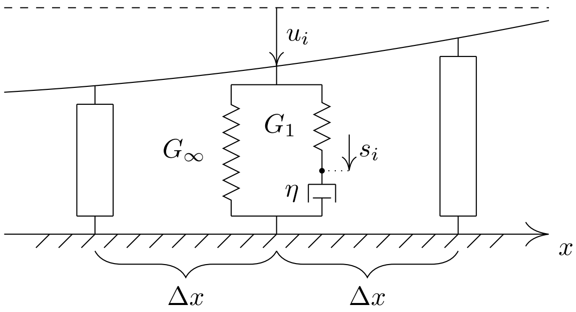

Figure 1. Displacements of the i-th rheologic element within the Winkler foundation.

the MDR-transformed profile of the sphere (in parabolic approximation) $g_i = x_i^2/R$:

$$u_i^j = d^j - \frac{x_i^2}{R}, \tag{23}$$

whereby the contact area in the compression phase is determined by the condition $u_i > 0$, in the restitution phase by the condition $q_i > 0$. The total force on the sphere can be determined by integration or, in the discrete case, summation of the line load, so that the equation of motion reads:

$$m\ddot{d}^j = -\sum_i q_i^j \Delta x. \tag{24}$$

With known line load q, equations (17) and (18) can now be evaluated. According to Willert (2020), the impact problem depends only on the two dimensionless parameters

$$\beta = \frac{G_\infty}{G_1}, \; \delta = \eta \left(\frac{R v_0}{m^2 G_\infty^3} \right)^{1/5}. \tag{25}$$

Figures 3 to 5 show the absolute pressure gradient in the half-space at three different points in time along the impact for the parameter combination $\beta = 0.1$ and $\delta = 0.3$, while figure 2 shows the evolution of the contact radius. For normalization the quantities of the fully elastic impact were used, i.e. the maximum contact radius $a_{\mathrm{max,e}}$, the impact duration T_{e} and the absolute pressure gradient in the center of the contact plane p_0':

$$a_{\mathrm{max,e}} = \left(\frac{15}{64} \frac{m v_0^2 R^2}{G_\infty} \right)^{1/5}, \; T_{\mathrm{e}} \approx 1.647 \left(\frac{m^2}{R v_0 G_\infty^2} \right)^{1/5}, \; p_0' = \frac{4 G_\infty}{R}. \tag{26}$$

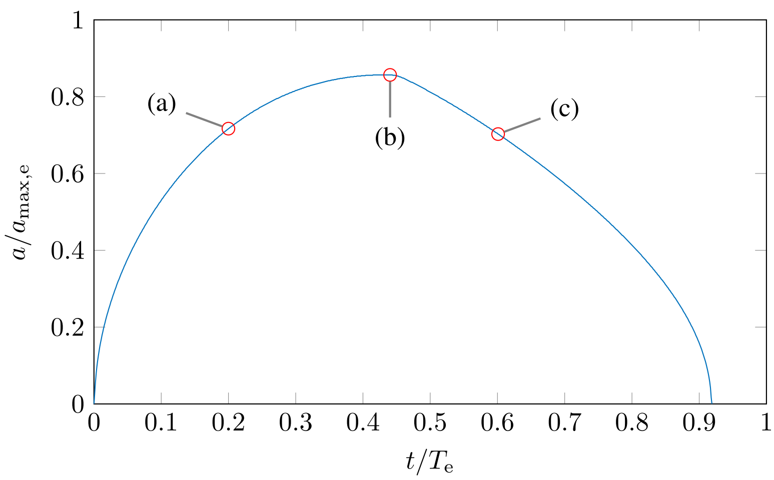

Figure 2. Contact radius $a(t)$ along the impact.

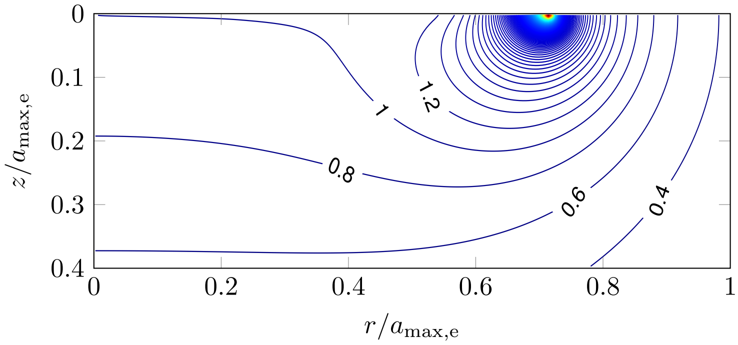

Figure 3. (a) Normalized absolute value of the hydrostatic pressure gradient $\left|\vec{\nabla}p_{\mathrm{hs}}\right|/p_0'$ at time $t/T_{\mathrm{e}} = 0.2$.

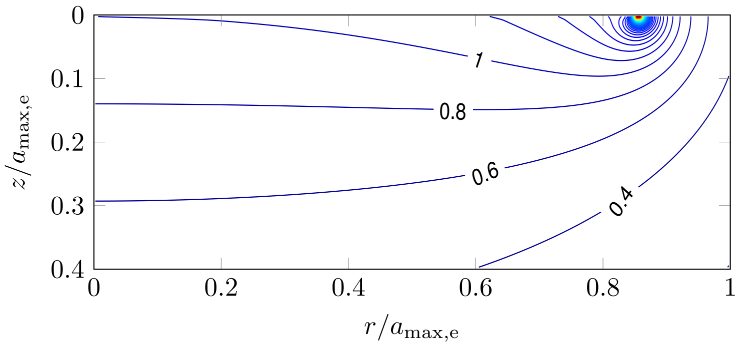

Figure 4. (b) Normalized absolute value of the hydrostatic pressure gradient $\left|\vec{\nabla}p_{\mathrm{hs}}\right|/p_0'$ at the time of maximum contact radius $t/T_{\mathrm{e}} \approx 0.432$.

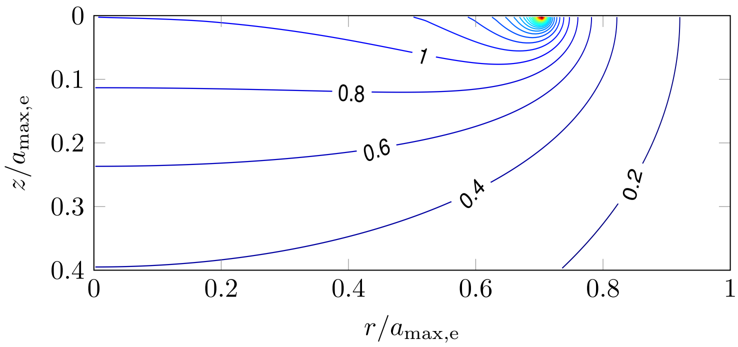

Figure 5. (c) Normalized absolute value of the hydrostatic pressure gradient $\left|\vec{\nabla}p_{\mathrm{hs}}\right|/p_0'$ at time $t/T_{\mathrm{e}} = 0.6$.

5 Conclusion

We presented two equations (8) and (15) to determine the stress state beneath the surface of a linear viscoelastic incompressible half-space in normal contact with an axisymmetric rigid indenter by superposition of elastic flat punch or Hertzian solutions along the line load from the MDR formalism. The validity of the flat punch superposition (from which the superposition of Hertzian solutions can be derived) was shown by comparing it with the solution for the stresses in a viscoelastic half-space given by Graham (1965); an analogous proof can be given by comparison with Ting's (1968) solution for the general case of arbitrary contact histories. The authors see the great advantage of the presented superposition principle in the elimination of (multiple) convolutions in the time domain (and with that the elimination of the need for knowledge about the entire contact history) due to the fact that in the framework of the MDR the line load can be determined from the equilibrium of rheological elements. This advantage arises especially in the numerical application of the viscoelastic MDR and this method.

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